Exercise 4

Consider the diffusion equation $u_t = u_{xx}$ in $\{0 < x < 1,\ 0 < t < \infty\}$ with $u(0,t) = u(1,t) = 0$ and $u(x,0) = 4x(1-x)$.

(a) Show that $0 < u(x,t) < 1$ for all $t > 0$ and $0 < x < 1$.

(b) Show that $u(x,t) = u(1-x,t)$ for all $t \geq 0$ and $0 \leq x \leq 1$.

(c) Use the energy method to show that $\int_0^1 u^2\,dx$ is a strictly decreasing function of t.

Solution

Part (a)

According to the minimum principle, the minimum of u must occur initially or on the boundary. The lowest value u takes is on the boundaries, where $u(0,t) = u(1,t) = 0$. The maximum principle says the maximum must occur initially or on the boundary. The highest value occurs initially, where $u(x = 1/2, 0) = 1$. Therefore, the solution remains bounded between 0 and 1 for all $t > 0$:

$$0 < u(x,t) < 1.$$

Part (b)

If we can show that $u(1-x,t)$ satisfies the Dirichlet problem, then $u(x,t)$ must be equal to $u(1-x,t)$ for all $0 \leq x \leq 1$ and $t \geq 0$ because the solution to the Dirichlet problem is unique. Start by taking derivatives of this function and using the chain rule.

$$\begin{aligned}
&\frac{\partial}{\partial t}u(1-x,t) = u_t \\
&\frac{\partial}{\partial x}u(1-x,t) = (-1)u_x = -u_x \\
&\frac{\partial^2}{\partial x^2}u(1-x,t) = (-1)(-1)u_{xx} = u_{xx}
\end{aligned}$$

Since $u_t = u_{xx}$,

$$\frac{\partial}{\partial t}u(1-x,t) = \frac{\partial^2}{\partial x^2}u(1-x,t),$$

which means $u(1-x,t)$ satisfies the diffusion equation. If we substitute $1-x$ for x in the initial and boundary conditions, we get

$$\begin{aligned}
0 < x < 1 \quad &\to \quad 0 < 1-x < 1 \quad \to \quad -1 < -x < 0 \quad \to \quad 0 < x < 1 \\
u(0,t) = 0 \quad &\to \quad u(1-0,t) = 0 \quad \to \quad u(1,t) = 0 \\
u(1,t) = 0 \quad &\to \quad u(1-1,t) = 0 \quad \to \quad u(0,t) = 0 \\
u(x,0) = 4x(1-x) \quad &\to \quad u(1-x,t) = 4(1-x)x
\end{aligned}$$

Hence, all conditions are the same for $u(1-x,t)$ as they are for $u(x,t)$. Therefore, because the solution to the Dirichlet problem is unique, $u(x,t) = u(1-x,t)$ for all $0 \leq x \leq 1$ and $t \geq 0$.

Part (c)

Begin by writing the diffusion equation.

$$u_t = u_{xx}$$

Multiply both sides by u.

$$uu_t = uu_{xx}$$

Rewrite both sides as follows.

$$\frac{1}{2}(u^2)_t = (u_x u)_x - u_x^2$$

Now integrate both sides over the length of the rod.

$$\int_0^1 \frac{1}{2}(u^2)_t\,dx = \int_0^1 \left[(u_x u)_x - u_x^2\right]\,dx$$

$$\frac{1}{2}\int_0^1 (u^2)_t\,dx = \int_0^1 (u_x u)_x\,dx - \int_0^1 u_x^2\,dx$$

$$\frac{1}{2}\frac{d}{dt}\int_0^1 u^2\,dx = u_x u\big|_0^1 - \int_0^1 u_x^2\,dx$$

$$\frac{1}{2}\frac{d}{dt}\int_0^1 u^2\,dx = [u_x(1,t)\underbrace{u(1,t)}_{=0} - u_x(0,t)\underbrace{u(0,t)}_{=0}] - \int_0^1 u_x^2\,dx$$

Hence,

$$\frac{d}{dt}\int_0^1 u^2\,dx = -2\underbrace{\int_0^1 u_x^2\,dx}_{\text{positive}}.$$

Therefore,

$$\int_0^1 u^2\,dx$$

is a strictly decreasing function of t.