Exercise 5

The purpose of this exercise is to show that the maximum principle is not true for the equation $u_t = xu_{xx}$, which has a variable coefficient.

(a) Verify that $u = -2xt - x^2$ is a solution. Find the location of its maximum in the closed rectangle $\{-2 \le x \le 2,\ 0 \le t \le 1\}$.

(b) Where precisely does our proof of the maximum principle break down for this equation?

Solution

Part (a)

Start by taking derivatives of the function, $u(x,t) = -2xt - x^2$.

$$\begin{aligned} u_t &= -2x \\ u_x &= -2t - 2x \\ u_{xx} &= -2 \end{aligned}$$

Hence, $u_t = xu_{xx}$, which means $u(x,t) = -2xt - x^2$ is a solution. Recall from calculus that to find the absolute maximum and minimum values of a continuous function u on a closed, bounded set D, there are three steps:

1. Find the values of u at the critical points of u in D.
2. Find the extreme values of u on the boundary of D.
3. The largest of the values from steps 1 and 2 is the absolute maximum value; the smallest of these values is the absolute minimum value.

The critical points of u occur where the first partial derivatives of u, u_t and u_x, equal 0, that is, when $x = 0$ and $t = 0$. The value of u here is

$$u(0,0) = 0.$$

Now we evaluate u along the boundary of the domain, $\{-2 \le x \le 2,\ 0 \le t \le 1\}$.

$u(-2,t) = 4t - 4$	$\rightarrow$	Lowest value ($t = 0$): -4,	Highest value ($t = 1$): 0
$u(2,t) = -4t - 4$	$\rightarrow$	Lowest value ($t = 1$): -8,	Highest value ($t = 0$): -4
$u(x,0) = -x^2$	$\rightarrow$	Lowest value ($x = \pm 2$): -4,	Highest value ($x = 0$): 0
$u(x,1) = -2x - x^2$	$\rightarrow$	Lowest value ($x = 2$): -8	

To find the highest value of $u(x,1)$, take the derivative of it with respect to x and set it equal to 0.

$$\frac{d}{dx}u(x,1) = -2 - 2x = 0,$$

which means $x = -1$ is where $u(x,1)$ has its highest value. The value of u here is

$$u(-1,1) = 1.$$

This is the highest value obtained in both steps; thus, $u = 1$ is the absolute maximum, and its location is the one we care about. Therefore, $x = -1$ and $t = 1$ is where the maximum occurs in the closed rectangle, $\{-2 \le x \le 2,\ 0 \le t \le 1\}$.

Part (b)

The diffusion inequality no longer holds because the coefficient of u_{xx} is not constant.

$$v_t - xv_{xx} = u_t - x(u + \varepsilon x^2)_{xx} = \underbrace{u_t - xu_{xx}}_{=0} - 2\varepsilon x = -2\varepsilon x,$$

which is not necessarily less than 0 since x can be negative. Thus, the existence of a maximum is not guaranteed.