Exercise 8

Consider the diffusion equation on $(0, l)$ with the Robin boundary conditions $u_x(0,t) - a_0 u(0,t) = 0$ and $u_x(l,t) + a_l u(l,t) = 0$. If $a_0 > 0$ and $a_l > 0$, use the energy method to show that the endpoints contribute to the decrease of $\int_0^l u^2(x,t)\,dx$. (This is interpreted to mean that part of the "energy" is lost at the boundary, so we call the boundary conditions "*radiating*" or "*dissipative.*")

Solution

The boundary conditions are

$$\begin{aligned} u_x(0,t) - a_0 u(0,t) = 0 \quad &\rightarrow \quad u_x(0,t) = a_0 u(0,t) \\ u_x(l,t) + a_l u(l,t) = 0 \quad &\rightarrow \quad u_x(l,t) = -a_l u(l,t). \end{aligned}$$

Begin by writing the diffusion equation.

$$u_t = k u_{xx}$$

Multiply both sides by u.

$$u u_t = k u u_{xx}$$

Rewrite both sides as follows.

$$\frac{1}{2}(u^2)_t = k[(u_x u)_x - u_x^2]$$

Now integrate both sides over the length of the rod.

$$\int_0^l \frac{1}{2}(u^2)_t\,dx = \int_0^l k[(u_x u)_x - u_x^2]\,dx$$

$$\frac{1}{2}\int_0^l \frac{\partial}{\partial t}(u^2)\,dx = k\left[\int_0^l (u_x u)_x\,dx - \int_0^l u_x^2\,dx\right]$$

$$\frac{1}{2}\frac{d}{dt}\int_0^l u^2\,dx = k\left[u_x u|_0^l - \int_0^l u_x^2\,dx\right]$$

$$\frac{1}{2}\frac{d}{dt}\int_0^l u^2\,dx = k\left[u_x(l,t)u(l,t) - u_x(0,t)u(0,t) - \int_0^l u_x^2\,dx\right]$$

Now multiply both sides by 2 and substitute the boundary conditions.

$$\frac{d}{dt}\int_0^l u^2\,dx = 2k\left\{-a_l[u(l,t)]^2 - a_0[u(0,t)]^2 - \int_0^l u_x^2\,dx\right\}$$

Because a_0 and a_l are positive, all terms on the right-hand side are negative. Therefore, the endpoints contribute to the decrease of

$$\int_0^l u^2\,dx,$$

and it is a strictly decreasing function of t.