Exercise 1

Consider the solution $1 - x^2 - 2kt$ of the diffusion equation. Find the locations of its maximum and its minimum in the closed rectangle $\{0 \leq x \leq 1,\ 0 \leq t \leq T\}$.

Solution

The Quick Way

The maximum occurs when x and t are as small as possible, that is, when $x = 0$ and $t = 0$. The minimum occurs when x and t are as large as possible, that is, when $x = 1$ and $t = T$.

The Systematic Way

Recall from calculus that to find the absolute maximum and minimum values of a continuous function u on a closed, bounded set D, there are three steps:

1. Find the values of u at the critical points of u in D.
2. Find the extreme values of u on the boundary of D.
3. The largest of the values from steps 1 and 2 is the absolute maximum value; the smallest of these values is the absolute minimum value.

The critical points of $u(x,t) = 1 - x^2 - 2kt$ occur where the first partial derivatives of u, u_t and u_x, equal 0.

$$\begin{aligned} u_t &= -2k \\ u_x &= -2x \end{aligned}$$

Because $k > 0$, u_t is never zero, so there are no critical points for this function. Now we evaluate u along the boundary of the domain, $\{0 \leq x \leq 1,\ 0 \leq t \leq T\}$.

$u(0,t) = 1 - 2kt$	$\rightarrow$	Lowest value $(t = T)$: $1 - 2kT$,	Highest value $(t = 0)$: 1
$u(1,t) = -2kt$	$\rightarrow$	Lowest value $(t = T)$: $-2kT$,	Highest value $(t = 0)$: 0
$u(x,0) = 1 - x^2$	$\rightarrow$	Lowest value $(x = 1)$: 0,	Highest value $(x = 0)$: 1
$u(x,T) = 1 - x^2 - 2kT$	$\rightarrow$	Lowest value $(x = 1)$: $-2kT$,	Highest value $(x = 0)$: $1 - 2kT$

The highest value obtained is $u = 1$ at $x = 0$ and $t = 0$, and the lowest value obtained is $-2kT$ at $x = 1$ and $t = T$. Therefore, in the closed rectangle, $\{0 \leq x \leq 1,\ 0 \leq t \leq T\}$, the maximum is located at $x = 0$ and $t = 0$, and the minimum is located at $x = 1$ and $t = T$.