Exercise 1

Consider the solution $1 - x^2 - 2kt$ of the diffusion equation. Find the locations of its maximum and its minimum in the closed rectangle $\{0 \leq x \leq 1, \ 0 \leq t \leq T\}$.

Solution

The Quick Way

The maximum occurs when $x$ and $t$ are as small as possible, that is, when $x = 0$ and $t = 0$. The minimum occurs when $x$ and $t$ are as large as possible, that is, when $x = 1$ and $t = T$.

The Systematic Way

Recall from calculus that to find the absolute maximum and minimum values of a continuous function $u$ on a closed, bounded set $D$, there are three steps:

1. Find the values of $u$ at the critical points of $u$ in $D$.

2. Find the extreme values of $u$ on the boundary of $D$.

3. The largest of the values from steps 1 and 2 is the absolute maximum value; the smallest of these values is the absolute minimum value.

The critical points of $u(x, t) = 1 - x^2 - 2kt$ occur where the first partial derivatives of $u$, $u_t$ and $u_x$, equal 0.

\[
\begin{align*}
  u_t &= -2k \\
  u_x &= -2x
\end{align*}
\]

Because $k > 0$, $u_t$ is never zero, so there are no critical points for this function. Now we evaluate $u$ along the boundary of the domain, $\{0 \leq x \leq 1, \ 0 \leq t \leq T\}$.

\[
\begin{align*}
  u(0, t) &= 1 - 2kt \quad \rightarrow \quad \text{Lowest value (t = T): } 1 - 2kT, \quad \text{Highest value (t = 0): } 1 \\
  u(1, t) &= -2kt \quad \rightarrow \quad \text{Lowest value (t = T): } -2kT, \quad \text{Highest value (t = 0): } 0 \\
  u(x, 0) &= 1 - x^2 \quad \rightarrow \quad \text{Lowest value (x = 1): } 0, \quad \text{Highest value (x = 0): } 1 \\
  u(x, T) &= 1 - x^2 - 2kT \quad \rightarrow \quad \text{Lowest value (x = 1): } -2kT, \quad \text{Highest value (x = 0): } 1 - 2kT
\end{align*}
\]

The highest value obtained is $u = 1$ at $x = 0$ and $t = 0$, and the lowest value obtained is $-2kT$ at $x = 1$ and $t = T$. Therefore, in the closed rectangle, $\{0 \leq x \leq 1, \ 0 \leq t \leq T\}$, the maximum is located at $x = 0$ and $t = 0$, and the minimum is located at $x = 1$ and $t = T$. 

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