Exercise 3

Consider the diffusion equation \( u_t = u_{xx} \) in the interval \((0,1)\) with \( u(0, t) = u(1, t) = 0 \) and \( u(x, 0) = 1 - x^2 \). Note that this initial function does not satisfy the boundary condition at the left end, but that the solution will satisfy it for all \( t > 0 \).

(a) Show that \( u(x, t) > 0 \) at all interior points \( 0 < x < 1, \ 0 < t < \infty \).

(b) For each \( t > 0 \), let \( \mu(t) = \) the maximum of \( u(x, t) \) over \( 0 \leq x \leq 1 \). Show that \( \mu(t) \) is a decreasing (i.e. nonincreasing) function of \( t \). (Hint: Let the maximum occur at the point \( X(t) \), so that \( \mu(t) = u(X(t), t) \). Differentiate \( \mu(t) \), assuming that \( X(t) \) is differentiable.)

(c) Draw a rough sketch of what you think the solution looks like \((u \text{ versus } x)\) at a few times. (If you have appropriate software available, compute it.)

Solution

Part (a)

According to the minimum principle, the lowest value of \( u \) can only occur initially or on the boundary. Since \( u(x, 0) = 1 - x^2 \) and \( 0 < x < 1, \ u > 0 \) initially, On the boundary, \( u(0, t) = u(1, t) = 0, \) so \( u = 0 \) is the minimum value. Therefore, by the minimum principle, \( u > 0 \) at all interior points \((0 < x < 1)\) for \( 0 < t < \infty \).

Part (b)

The goal here is to show that the maximum of \( u, \mu(t) \), is a decreasing function of time, that is,

\[
\frac{d\mu}{dt} < 0
\]

for each \( t > 0 \). Following the hint, suppose the maximum occurs at the \( x \)-coordinate, \( X(t) \).

\[
\mu(t) = u(x = X(t), t)
\]

Take the derivative of \( \mu \) with respect to \( t \), using the chain rule since both arguments are functions of \( t \).

\[
\frac{d\mu}{dt} = \frac{\partial u}{\partial x} \frac{dx}{dt} + \frac{\partial u}{\partial t} \frac{dt}{dt} = u_x(X(t), t) \frac{dX}{dt} + u_t(X(t), t)
\]

Use the fact that \( u_t = u_{xx} \).

\[
\frac{d\mu}{dt} = u_x(X(t), t) \frac{dX}{dt} + u_{xx}(X(t), t)
\]

At the maximum the slope of \( u \) is zero and the concavity is downward, so \( u_x(X(t), t) = 0 \) and \( u_{xx}(X(t), t) < 0 \). Therefore,

\[
\frac{d\mu}{dt} < 0,
\]

which means \( \mu(t) \) is a decreasing function of \( t \).

\[^1\text{Special thanks to L. Baker for the correction in part (b).}\]

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Part (c)

The solution to the diffusion equation that satisfies the given boundary conditions and initial condition is

\[ u(x, t) = \sum_{n=1}^{\infty} A_n e^{-n^2 \pi^2 t} \sin n\pi x, \]

where

\[ A_n = \frac{2}{n^3 \pi^3} [2 - 2(-1)^n + n^2 \pi^2]. \]

Shown below are graphs of \( u \) as a function of \( x \) for five different times.

Figure 1: The concentration profile at \( t = 0 \).
Figure 2: The concentration profile at $t = 0.001$.

Figure 3: The concentration profile at $t = 0.01$. 

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Figure 4: The concentration profile at $t = 0.03$.

Figure 5: The concentration profile at $t = 0.1$. 