Exercise 6

Prove the *comparison principle* for the diffusion equation: If u and v are two solutions, and if $u \leq v$ for $t = 0$, for $x = 0$, and for $x = l$, then $u \leq v$ for $0 \leq t < \infty$, $0 \leq x \leq l$.

Solution

If u and v are solutions to the diffusion equation, then they both have to satisfy it.

$$v_t = kv_{xx} \tag{1}$$
$$u_t = ku_{xx} \tag{2}$$

We're told that $v \geq u$ initially and on the boundary. That is,

$$\begin{aligned} v(x,0) \geq u(x,0) \quad &\rightarrow \quad v(x,0) - u(x,0) \geq 0 \\ v(0,t) \geq u(0,t) \quad &\rightarrow \quad v(0,t) - u(0,t) \geq 0 \\ v(l,t) \geq u(l,t) \quad &\rightarrow \quad v(l,t) - u(l,t) \geq 0. \end{aligned}$$

Consider the difference between (1) and (2).

$$v_t - u_t = kv_{xx} - ku_{xx}$$

Now factor the equation.

$$(v-u)_t = k(v-u)_{xx}$$

Let $w = v - u$ so we get

$$w_t = kw_{xx}, \quad w(x,0) \geq 0,\ w(0,t) \geq 0,\ w(l,t) \geq 0.$$

According to the minimum principle, w will have a minimum either initially or on the boundary. Because the lowest value of w is 0, $w \geq 0$ for $0 \leq t < \infty$ and $0 \leq x \leq l$. Therefore, $v \geq u$ for $0 \leq t < \infty$ and $0 \leq x \leq l$.