Exercise 1

Solve the diffusion equation with the initial condition

$$\phi(x) = 1 \quad \text{for } |x| < l \qquad \text{and} \qquad \phi(x) = 0 \quad \text{for } |x| > l.$$

Write your answer in terms of $\mathscr{E}\text{rf}(x)$.

Solution

Solution by the Similarity Method

We have to solve the initial value problem,

$$u_t = ku_{xx}, \quad u(x, 0) = \phi(x). \tag{1}$$

In order to do so, we'll solve for the Green's function $G(x, t)$ in the corresponding PDE,

$$G_t = kG_{xx}, \quad G(x, 0) = \delta(x), \tag{2}$$

where $\delta(x)$, the Dirac delta function, is defined as

$$\delta(x) = \begin{cases} 1 & x = 0 \\ 0 & x \neq 0 \end{cases}.$$

The reason we're solving the equation with the delta function is that it has the extremely useful "sifting" property,

$$\int_{-\infty}^{\infty} f(s)\delta(x - s)\, ds = f(x),$$

so the solution to the initial value problem (1) in terms of the Green's function is

$$u(x, t) = \int_{-\infty}^{\infty} G(x - s, t)\phi(s)\, ds.$$

This can be verified by substituting this form for u into (1). Now we will go about solving (2) for $G(x, t)$ by using the similarity method (also known as the combination of variables method). Because u is a dimensionless quantity (that is, it yields a pure number with no units) the variables x, t, and k have to appear in the solution in a dimensionless combination. x has units of meters, t has units of seconds, and k has units of meters^2/second, so the combination of variables has to be

$$\frac{x^2}{kt}$$

or any convenient multiple or power thereof. Therefore,

$$u = u(\eta), \quad \text{where we choose } \eta = \frac{x}{\sqrt{kt}}.$$

We choose this particular form for η so the process of getting the final answer is smoother. We're trying to solve (2) for G, though, and G is not dimensionless; as can be seen from the initial condition, it has the same dimensions as $\delta(x)$. $\delta(x)$ has the inverse dimension of its argument, so G has dimensions of meters^{-1}. Thus, G has to be of the form,

$$G(x,t) = \frac{1}{\sqrt{kt}} g\left(\frac{x}{\sqrt{kt}}\right),$$

where g is an arbitrary function. In order to determine g, we have to plug this form into (2) and solve the resulting ODE. To start, write the expressions for G_t and G_{xx}.

$$\begin{aligned} \frac{\partial G}{\partial t} &= -\frac{1}{2\sqrt{kt^3}} g + \frac{1}{\sqrt{kt}}\left(-\frac{x}{2\sqrt{kt^3}}\right) g' \\ \frac{\partial G}{\partial x} &= \frac{1}{\sqrt{kt}} \cdot \frac{1}{\sqrt{kt}} g' = \frac{1}{kt} g' \\ \frac{\partial^2 G}{\partial x^2} &= \frac{1}{kt} \cdot \frac{1}{\sqrt{kt}} g'' = \frac{1}{\sqrt{k^3 t^3}} g'' \end{aligned}$$

Substituting these expressions into (2) gives

$$-\frac{1}{2\sqrt{kt^3}} g - \frac{1}{2\sqrt{kt^3}} \cdot \frac{x}{\sqrt{kt}} g' = \frac{1}{\sqrt{kt^3}} g''.$$

Cancel common terms and move everything to one side.

$$g'' + \frac{1}{2}\frac{x}{\sqrt{kt}} g' + \frac{1}{2} g = 0$$

Use the combination variable η.

$$g'' + \frac{\eta}{2} g' + \frac{1}{2} g = 0$$

The last two terms on the left side can be written as one using the product rule.

$$g'' + \left(\frac{\eta}{2} g\right)' = 0$$

Integrate both sides of the equation.

$$g' + \frac{\eta}{2} g = C_1$$

This is an inhomogeneous first-order linear differential equation that can be solved with an integrating factor. The integrating factor is

$$I = e^{\int \frac{\eta}{2}\, d\eta} = e^{\frac{\eta^2}{4}}.$$

Multiply both sides by I.

$$e^{\frac{\eta^2}{4}} g' + \frac{\eta}{2} e^{\frac{\eta^2}{4}} g = C_1 e^{\frac{\eta^2}{4}}$$

The two terms on the left side can be written as one using the product rule.

$$\left(e^{\frac{\eta^2}{4}} g\right)' = C_1 e^{\frac{\eta^2}{4}}$$

Integrate both sides of the equation a second time.

$$e^{\frac{\eta^2}{4}} g = \int^{\eta} C_1 e^{\frac{s^2}{4}}\, ds + C_2$$

Hence, the arbitrary function g is

$$g(\eta) = e^{-\frac{\eta^2}{4}} \left[C_1 \int^{\eta} e^{\frac{s^2}{4}}\, ds + C_2 \right],$$

and consequently, the Green's function is

$$G = \frac{1}{\sqrt{kt}} g(\eta) = \frac{e^{-\frac{\eta^2}{4}}}{\sqrt{kt}} \left[C_1 \int^{\eta} e^{\frac{s^2}{4}}\, ds + C_2 \right]. \tag{3}$$

The next order of business is to determine the constants of integration, C_1 and C_2. We need to return to the diffusion equation and the initial condition in (2) to figure these out.

$$G_t = kG_{xx}$$

Integrate both sides of the equation with respect to x over the whole line.

$$\int_{-\infty}^{\infty} G_t\, dx = \int_{-\infty}^{\infty} kG_{xx}\, dx$$

Take out the time derivative from the left side and evaluate the right side.

$$\frac{d}{dt} \int_{-\infty}^{\infty} G\, dx = kG_x|_{-\infty}^{\infty}$$

We assume that G and G_x tend to 0 as $x \to \pm\infty$, so we have

$$\frac{d}{dt} \int_{-\infty}^{\infty} G\, dx = 0.$$

This implies that the quantity,

$$\int_{-\infty}^{\infty} G\, dx,$$

remains constant for all time. Initially $G(x, 0) = \delta(x)$, so

$$\int_{-\infty}^{\infty} G(x,t)\, dx = \int_{-\infty}^{\infty} G(x,0)\, dx = \int_{-\infty}^{\infty} \delta(x)\, dx = 1. \tag{4}$$

In order for this integral to converge, C_1 has to be 0. In terms of x and t, (3) becomes

$$G(x,t) = \frac{C_2}{\sqrt{kt}} e^{-\frac{x^2}{4kt}}.$$

C_2 can be thought of as a normalization constant that we determine by plugging into (4).

$$\int_{-\infty}^{\infty} G(x,t)\, dx = \int_{-\infty}^{\infty} \frac{C_2}{\sqrt{kt}} e^{-\frac{x^2}{4kt}}\, dx = 1$$

Make the following substitution to solve the integral.

$$v = \frac{x}{\sqrt{4kt}}$$
$$dv = \frac{dx}{\sqrt{4kt}} \quad \rightarrow \quad 2\, dv = \frac{dx}{\sqrt{kt}}$$

The integral becomes

$$2C_2 \int_{-\infty}^{\infty} e^{-v^2}\, dv = 1,$$

and it evaluates to $\sqrt{\pi}$.

$$2C_2\sqrt{\pi} = 1$$

Solving for C_2 yields

$$C_2 = \frac{1}{\sqrt{4\pi}}.$$

Therefore, the Green's function is

$$G(x,t) = \frac{1}{\sqrt{4\pi kt}} e^{-\frac{x^2}{4kt}},$$

and the solution to the initial value problem in (1) is

$$u(x,t) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi kt}} e^{-\frac{(x-s)^2}{4kt}} \phi(s)\, ds. \tag{5}$$

$u(x,t)$ can be interpreted as the convolution of the initial condition with a Gaussian filter. At every point x, $u(x,t)$ is an averaged, or smoothed, version of the initial condition over an interval of width $\sqrt{kt}$. As t increases, the range of the filter grows and $u(x,t)$ becomes increasingly smooth over x. Any discontinuities or kinks that are present in the initial condition are smoothed out. In this exercise, the initial condition is

$$\phi(x) = \begin{cases} 1 & |x| < l \\ 0 & |x| > l \end{cases}.$$

If we substitute this into the formula in (5), then we get

$$u(x,t) = \int_{-l}^{l} \frac{1}{\sqrt{4\pi kt}} e^{-\frac{(x-s)^2}{4kt}}\, ds.$$

Make the following substitution to solve the integral.

$$v = \frac{x-s}{\sqrt{4kt}}$$

$$dv = -\frac{ds}{\sqrt{4kt}} \quad \rightarrow \quad -dv = \frac{ds}{\sqrt{4kt}}$$

The integral becomes

$$u(x,t) = -\int_{\frac{x+l}{\sqrt{4kt}}}^{\frac{x-l}{\sqrt{4kt}}} \frac{1}{\sqrt{\pi}} e^{-v^2}\, dv.$$

Bring the constant out in front and switch the limits of integration to eliminate the minus sign.

$$u(x,t) = \frac{1}{\sqrt{\pi}} \int_{\frac{x-l}{\sqrt{4kt}}}^{\frac{x+l}{\sqrt{4kt}}} e^{-v^2}\, dv$$

Split up the integral into two.

$$u(x,t) = \frac{1}{\sqrt{\pi}} \int_{\frac{x-l}{\sqrt{4kt}}}^{0} e^{-v^2}\, dv + \frac{1}{\sqrt{\pi}} \int_{0}^{\frac{x+l}{\sqrt{4kt}}} e^{-v^2}\, dv$$

Switch the limits of the first integral and add a minus sign in front of it.

$$u(x,t) = -\frac{1}{\sqrt{\pi}} \int_0^{\frac{x-l}{\sqrt{4kt}}} e^{-v^2}\, dv + \frac{1}{\sqrt{\pi}} \int_0^{\frac{x+l}{\sqrt{4kt}}} e^{-v^2}\, dv$$

The error function, $\operatorname{erf} z$, is defined as

$$\operatorname{erf} z = \frac{2}{\sqrt{\pi}} \int_0^z e^{-v^2}\, dv,$$

so

$$u(x,t) = -\frac{1}{2} \operatorname{erf}\left(\frac{x-l}{\sqrt{4kt}}\right) + \frac{1}{2} \operatorname{erf}\left(\frac{x+l}{\sqrt{4kt}}\right).$$

Therefore,

$$u(x,t) = \frac{1}{2}\left[\operatorname{erf}\left(\frac{x+l}{\sqrt{4kt}}\right) - \operatorname{erf}\left(\frac{x-l}{\sqrt{4kt}}\right)\right].$$

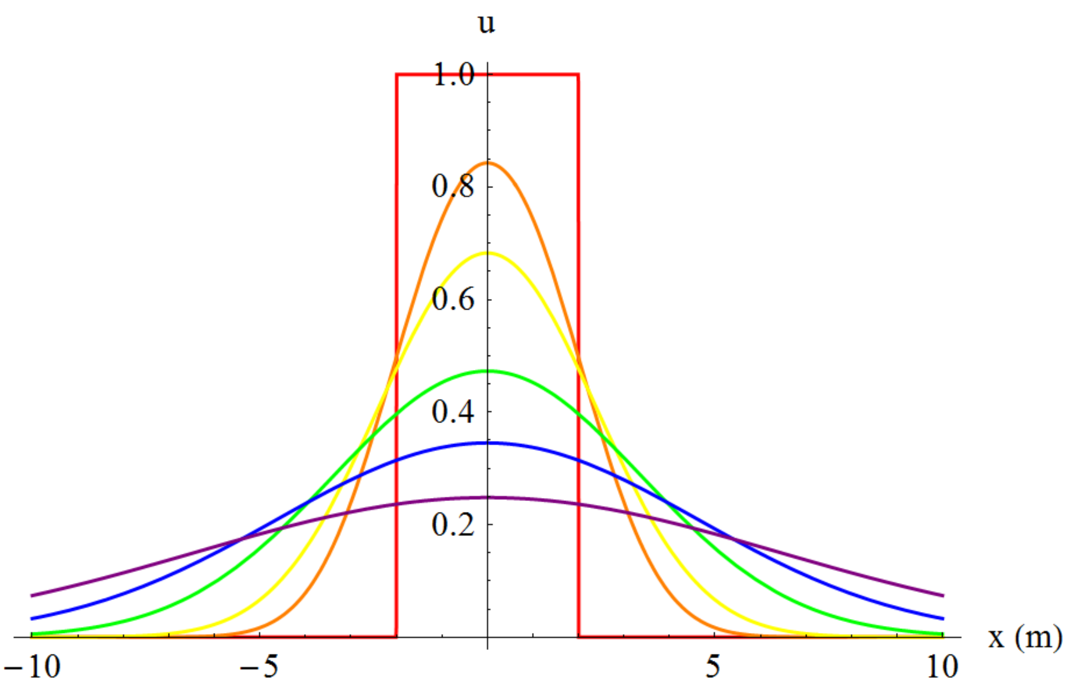

Figure 1: Plot of the solution $u(x,t)$ with $k = 1$ m^2/s and $l = 2$ m for six different times: $t = 0$ s (red), $t = 1$ s (orange), $t = 2$ s (yellow), $t = 5$ s (green), $t = 10$ s (blue), and $t = 20$ s (purple).

Solution by Exploitation of the Invariance Properties

The initial condition,

$$\phi(x) = \begin{cases} 1 & |x| < l \\ 0 & |x| > l \end{cases},$$

can be written in terms of the Heaviside function, which is defined as

$$H(x) = \begin{cases} 1 & x > 0 \\ 0 & x < 0 \end{cases},$$

as

$$\phi(x) = H(x + l) - H(x - l).$$

As Strauss derived in the text, the solution to the diffusion equation with $H(x)$ as the initial condition is

$$Q(x,t) = \frac{1}{2} + \frac{1}{\sqrt{\pi}} \int_0^{\frac{x}{\sqrt{4kt}}} e^{-p^2}\, dp = \frac{1}{2} + \frac{1}{2} \operatorname{erf}\left(\frac{x}{\sqrt{4kt}}\right).$$

Since any translate of a solution is a solution to the diffusion equation and any linear combination of solutions is a solution to the diffusion equation, we can write down the solution with $\phi(x)$ for the initial condition immediately.

$$u(x,t) = Q(x + l, t) - Q(x - l, t)$$

Substituting the expression for Q yields the following.

$$u(x,t) = \left[\cancel{\frac{1}{2}} + \frac{1}{2} \operatorname{erf}\left(\frac{x+l}{\sqrt{4kt}}\right)\right] - \left[\cancel{\frac{1}{2}} + \frac{1}{2} \operatorname{erf}\left(\frac{x-l}{\sqrt{4kt}}\right)\right]$$

Simplifying gives us the same result as before.

$$u(x,t) = \frac{1}{2}\left[\operatorname{erf}\left(\frac{x+l}{\sqrt{4kt}}\right) - \operatorname{erf}\left(\frac{x-l}{\sqrt{4kt}}\right)\right]$$

Note that the Heaviside function and the Dirac delta function are related by

$$\frac{dH}{dx} = \delta(x),$$

so we could derive the formula for $Q(x,t)$ by integrating the result we got for $G(x,t)$ since any integral of a solution to the diffusion equation is also a solution.

$$Q(x,t) = \int_0^x G(s,t)\, ds + D = \frac{1}{\sqrt{4\pi kt}} \int_0^x e^{-\frac{s^2}{4kt}}\, ds + D$$

Make the following substitution to solve the integral.

$$p = \frac{s}{\sqrt{4kt}} \quad \rightarrow \quad dp = \frac{ds}{\sqrt{4kt}}$$

The equation becomes

$$Q(x,t) = \frac{1}{\sqrt{\pi}} \int_0^{\frac{x}{\sqrt{4kt}}} e^{-p^2}\, dp + D = \frac{1}{2} \operatorname{erf}\left(\frac{x}{\sqrt{4kt}}\right) + D.$$

Since we need Q to emulate the Heaviside function as $t \to 0$, $D = 1/2$.