Exercise 14

Let $\phi(x)$ be a continuous function such that $|\phi(x)| \leq Ce^{ax^2}$. Show that formula (8) for the solution of the diffusion equation makes sense for $0 < t < 1/(4ak)$, but not necessarily for larger t.

Solution

The solution to the diffusion equation, $u_t = ku_{xx}$, is given by formula (8) as

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-\frac{(x-s)^2}{4kt}} \phi(s)\, ds. \tag{8}$$

The fact that $|\phi(x)| \leq Ce^{ax^2}$ prompts us to consider the magnitude of $u(x,t)$.

$$|u(x,t)| = \frac{1}{\sqrt{4\pi kt}} \left| \int_{-\infty}^{\infty} e^{-\frac{(x-s)^2}{4kt}} \phi(s)\, ds \right|$$

We can bring the absolute value sign to the integrand, provided we make this an inequality.

$$|u(x,t)| \leq \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} \left| e^{-\frac{(x-s)^2}{4kt}} \phi(s) \right| ds$$

The exponential function is never negative, so we can remove the absolute values around it. It is here where we substitute $|\phi(x)| \leq Ce^{ax^2}$.

$$|u(x,t)| \leq \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} \left| e^{-\frac{(x-s)^2}{4kt}} \right| |\phi(s)|\, ds \leq \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-\frac{(x-s)^2}{4kt}} Ce^{as^2}\, ds$$

The problem statement says the integral won't necessarily make sense when $t = 1/(4ak)$, so let's make the substitution to see why.

$$t = \frac{1}{4ak} \quad \rightarrow \quad a = \frac{1}{4kt} \quad \rightarrow \quad \frac{a}{\pi} = \frac{1}{4\pi kt} \quad \rightarrow \quad \sqrt{\frac{a}{\pi}} = \frac{1}{\sqrt{4\pi kt}}$$

The integral becomes

$$\left| u\left(x, \frac{1}{4ak}\right) \right| \leq C\sqrt{\frac{a}{\pi}} \int_{-\infty}^{\infty} e^{-a(x-s)^2} e^{as^2}\, ds.$$

Combine the exponential functions.

$$\left| u\left(x, \frac{1}{4ak}\right) \right| \leq C\sqrt{\frac{a}{\pi}} \int_{-\infty}^{\infty} e^{-ax^2+2axs}\, ds$$

Pull the constant out in front of the integral and then proceed with the integration.

$$\left| u\left(x, \frac{1}{4ak}\right) \right| \leq C\sqrt{\frac{a}{\pi}} e^{-ax^2} \int_{-\infty}^{\infty} e^{2axs}\, ds = C\sqrt{\frac{a}{\pi}} e^{-ax^2} \left. \frac{e^{2axs}}{2ax} \right|_{-\infty}^{\infty} = C\sqrt{\frac{a}{\pi}} \frac{e^{-ax^2}}{2ax} \left(e^{\infty} - \frac{1}{e^{\infty}} \right)$$

Therefore,

$$\left| u\left(x, \frac{1}{4ak}\right) \right| \leq \infty.$$

What this indicates is that the integral solution given by (8) may or may not be bounded when $t = 1/(4ak)$. This is also the case for any later time. However, when t is any smaller than $1/(4ak)$, there is an e^{-s^2} term that makes the integral converge. Since the integral solution only holds for $t > 0$, formula (8) makes sense when $0 < t < 1/(4ak)$ but not necessarily for later times.