Exercise 2

Do the same for $\phi(x) = 1$ for $x > 0$ and $\phi(x) = 3$ for $x < 0$.

Solution

Solution by the Similarity Method

We have to solve the initial value problem,

$$u_t = ku_{xx}, \quad u(x,0) = \phi(x). \tag{1}$$

In order to do so, we'll solve for the Green's function $G(x,t)$ in the corresponding PDE,

$$G_t = kG_{xx}, \quad G(x,0) = \delta(x), \tag{2}$$

where $\delta(x)$, the Dirac delta function, is defined as

$$\delta(x) = \begin{cases} 1 & x = 0 \\ 0 & x \neq 0 \end{cases}.$$

The reason we're solving the equation with the delta function is that it has the extremely useful "sifting" property,

$$\int_{-\infty}^{\infty} f(s)\delta(x-s)\,ds = f(x),$$

so the solution to the initial value problem (1) in terms of the Green's function is

$$u(x,t) = \int_{-\infty}^{\infty} G(x-s,t)\phi(s)\,ds.$$

This can be verified by substituting this form for u into (1). Now we will go about solving (2) for $G(x,t)$ by using the similarity method (also known as the combination of variables method). Because u is a dimensionless quantity (that is, it yields a pure number with no units) the variables x, t, and k have to appear in the solution in a dimensionless combination. x has units of meters, t has units of seconds, and k has units of meters^2/second, so the combination of variables has to be

$$\frac{x^2}{kt}$$

or any convenient multiple or power thereof. Therefore,

$$u = u(\eta), \quad \text{where we choose } \eta = \frac{x}{\sqrt{kt}}.$$

We choose this particular form for η so the process of getting the final answer is smoother. We're trying to solve (2) for G, though, and G is not dimensionless; as can be seen from the initial condition, it has the same dimensions as $\delta(x)$. $\delta(x)$ has the inverse dimension of its argument, so G has dimensions of meters^{-1}. Thus, G has to be of the form,

$$G(x,t) = \frac{1}{\sqrt{kt}} g\left(\frac{x}{\sqrt{kt}}\right),$$

where g is an arbitrary function. In order to determine g, we have to plug this form into (2) and solve the resulting ODE. To start, write the expressions for G_t and G_{xx}.

$$\frac{\partial G}{\partial t} = -\frac{1}{2\sqrt{kt^3}}g + \frac{1}{\sqrt{kt}}\left(-\frac{x}{2\sqrt{kt^3}}\right)g'$$
$$\frac{\partial G}{\partial x} = \frac{1}{\sqrt{kt}} \cdot \frac{1}{\sqrt{kt}}g' = \frac{1}{kt}g'$$
$$\frac{\partial^2 G}{\partial x^2} = \frac{1}{kt} \cdot \frac{1}{\sqrt{kt}}g'' = \frac{1}{\sqrt{k^3t^3}}g''$$

Substituting these expressions into (2) gives

$$-\frac{1}{2\sqrt{kt^3}}g - \frac{1}{2\sqrt{kt^3}} \cdot \frac{x}{\sqrt{kt}}g' = \frac{1}{\sqrt{kt^3}}g''.$$

Cancel common terms and move everything to one side.

$$g'' + \frac{1}{2}\frac{x}{\sqrt{kt}}g' + \frac{1}{2}g = 0$$

Use the combination variable η.

$$g'' + \frac{\eta}{2}g' + \frac{1}{2}g = 0$$

The last two terms on the left side can be written as one using the product rule.

$$g'' + \left(\frac{\eta}{2}g\right)' = 0$$

Integrate both sides of the equation.

$$g' + \frac{\eta}{2}g = C_1$$

This is an inhomogeneous first-order linear differential equation that can be solved with an integrating factor. The integrating factor is

$$I = e^{\int \frac{\eta}{2}\,d\eta} = e^{\frac{\eta^2}{4}}.$$

Multiply both sides by I.

$$e^{\frac{\eta^2}{4}}g' + \frac{\eta}{2}e^{\frac{\eta^2}{4}}g = C_1 e^{\frac{\eta^2}{4}}$$

The two terms on the left side can be written as one using the product rule.

$$\left(e^{\frac{\eta^2}{4}}g\right)' = C_1 e^{\frac{\eta^2}{4}}$$

Integrate both sides of the equation a second time.

$$e^{\frac{\eta^2}{4}}g = \int^{\eta} C_1 e^{\frac{s^2}{4}}\,ds + C_2$$

Hence, the arbitrary function g is

$$g(\eta) = e^{-\frac{\eta^2}{4}}\left[C_1\int^{\eta} e^{\frac{s^2}{4}}\,ds + C_2\right],$$

and consequently, the Green's function is

$$G = \frac{1}{\sqrt{kt}} g(\eta) = \frac{e^{-\frac{\eta^2}{4}}}{\sqrt{kt}} \left[C_1 \int^{\eta} e^{\frac{s^2}{4}}\, ds + C_2 \right]. \tag{3}$$

The next order of business is to determine the constants of integration, C_1 and C_2. We need to return to the diffusion equation and the initial condition in (2) to figure these out.

$$G_t = kG_{xx}$$

Integrate both sides of the equation with respect to x over the whole line.

$$\int_{-\infty}^{\infty} G_t\, dx = \int_{-\infty}^{\infty} kG_{xx}\, dx$$

Take out the time derivative from the left side and evaluate the right side.

$$\frac{d}{dt} \int_{-\infty}^{\infty} G\, dx = kG_x|_{-\infty}^{\infty}$$

We assume that G and G_x tend to 0 as $x \to \pm\infty$, so we have

$$\frac{d}{dt} \int_{-\infty}^{\infty} G\, dx = 0.$$

This implies that the quantity,

$$\int_{-\infty}^{\infty} G\, dx,$$

remains constant for all time. Initially $G(x, 0) = \delta(x)$, so

$$\int_{-\infty}^{\infty} G(x,t)\, dx = \int_{-\infty}^{\infty} G(x,0)\, dx = \int_{-\infty}^{\infty} \delta(x)\, dx = 1. \tag{4}$$

In order for this integral to converge, C_1 has to be 0. In terms of x and t, (3) becomes

$$G(x,t) = \frac{C_2}{\sqrt{kt}} e^{-\frac{x^2}{4kt}}.$$

C_2 can be thought of as a normalization constant that we determine by plugging into (4).

$$\int_{-\infty}^{\infty} G(x,t)\, dx = \int_{-\infty}^{\infty} \frac{C_2}{\sqrt{kt}} e^{-\frac{x^2}{4kt}}\, dx = 1$$

Make the following substitution to solve the integral.

$$v = \frac{x}{\sqrt{4kt}}$$

$$dv = \frac{dx}{\sqrt{4kt}} \quad \to \quad 2\, dv = \frac{dx}{\sqrt{kt}}$$

The integral becomes

$$2C_2 \int_{-\infty}^{\infty} e^{-v^2}\, dv = 1,$$

and it evaluates to $\sqrt{\pi}$.

$$2C_2\sqrt{\pi} = 1$$

Solving for C_2 yields

$$C_2 = \frac{1}{\sqrt{4\pi}}.$$

Therefore, the Green's function is

$$G(x,t) = \frac{1}{\sqrt{4\pi kt}} e^{-\frac{x^2}{4kt}},$$

and the solution to the initial value problem in (1) is

$$u(x,t) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi kt}} e^{-\frac{(x-s)^2}{4kt}} \phi(s)\, ds. \tag{5}$$

$u(x,t)$ can be interpreted as the convolution of the initial condition with a Gaussian filter. At every point x, $u(x,t)$ is an averaged, or smoothed, version of the initial condition over an interval of width $\sqrt{kt}$. As t increases, the range of the filter grows and $u(x,t)$ becomes increasingly smooth over x. Any discontinuities or kinks that are present in the initial condition are smoothed out. In this exercise, the initial condition is

$$\phi(x) = \begin{cases} 1 & x > 0 \\ 3 & x < 0 \end{cases}.$$

If we substitute this into the formula in (5), then we get

$$u(x,t) = \int_{-\infty}^{0} \frac{1}{\sqrt{4\pi kt}} e^{-\frac{(x-s)^2}{4kt}} 3\, ds + \int_{0}^{\infty} \frac{1}{\sqrt{4\pi kt}} e^{-\frac{(x-s)^2}{4kt}}\, ds.$$

Make the following substitutions to solve the integrals.

$$v = \frac{x-s}{\sqrt{4kt}} \qquad\qquad q = \frac{s-x}{\sqrt{4kt}}$$

$$dv = -\frac{ds}{\sqrt{4kt}} \quad \to \quad -dv = \frac{ds}{\sqrt{4kt}} \qquad\qquad dq = \frac{ds}{\sqrt{4kt}}$$

The integrals become

$$u(x,t) = -\int_{\infty}^{\frac{x}{\sqrt{4kt}}} \frac{3}{\sqrt{\pi}} e^{-v^2}\, dv + \int_{-\frac{x}{\sqrt{4kt}}}^{\infty} \frac{1}{\sqrt{\pi}} e^{-q^2}\, dq.$$

Bring the constants out in front and switch the limits of integration to eliminate the minus sign.

$$u(x,t) = \frac{1}{\sqrt{\pi}} \left[3\int_{\frac{x}{\sqrt{4kt}}}^{\infty} e^{-v^2}\, dv + \int_{-\frac{x}{\sqrt{4kt}}}^{\infty} e^{-q^2}\, dq \right]$$

Split up the integrals now to get the appropriate limits.

$$u(x,t) = \frac{1}{\sqrt{\pi}} \left[3\int_{0}^{\infty} e^{-v^2}\, dv - 3\int_{0}^{\frac{x}{\sqrt{4kt}}} e^{-v^2}\, dv + \int_{-\frac{x}{\sqrt{4kt}}}^{0} e^{-q^2}\, dq + \int_{0}^{\infty} e^{-q^2}\, dq \right]$$

Make the substitution $p = -q$ and $dp = -dq$ in the third integral and combine the first and last integrals. We can do this because v and q are just dummy variables.

$$u(x,t) = \frac{1}{\sqrt{\pi}}\left[4\int_0^\infty e^{-v^2}\,dv - 3\int_0^{\frac{x}{\sqrt{4kt}}} e^{-v^2}\,dv + \int_0^{\frac{x}{\sqrt{4kt}}} e^{-p^2}\,dp\right]$$

Combine the last integrals. p and v are just dummy variables as well.

$$u(x,t) = \frac{1}{\sqrt{\pi}}\left[4\int_0^\infty e^{-v^2}\,dv - 2\int_0^{\frac{x}{\sqrt{4kt}}} e^{-v^2}\,dv\right]$$

The integral on the left evaluates to $\sqrt{\pi}/2$.

$$u(x,t) = \frac{1}{\sqrt{\pi}}\left[4\cdot\frac{\sqrt{\pi}}{2} - 2\int_0^{\frac{x}{\sqrt{4kt}}} e^{-v^2}\,dv\right]$$

The error function, erf z, is defined as

$$\operatorname{erf} z = \frac{2}{\sqrt{\pi}}\int_0^z e^{-v^2}\,dv,$$

so

$$u(x,t) = 2 - \operatorname{erf}\left(\frac{x}{\sqrt{4kt}}\right).$$

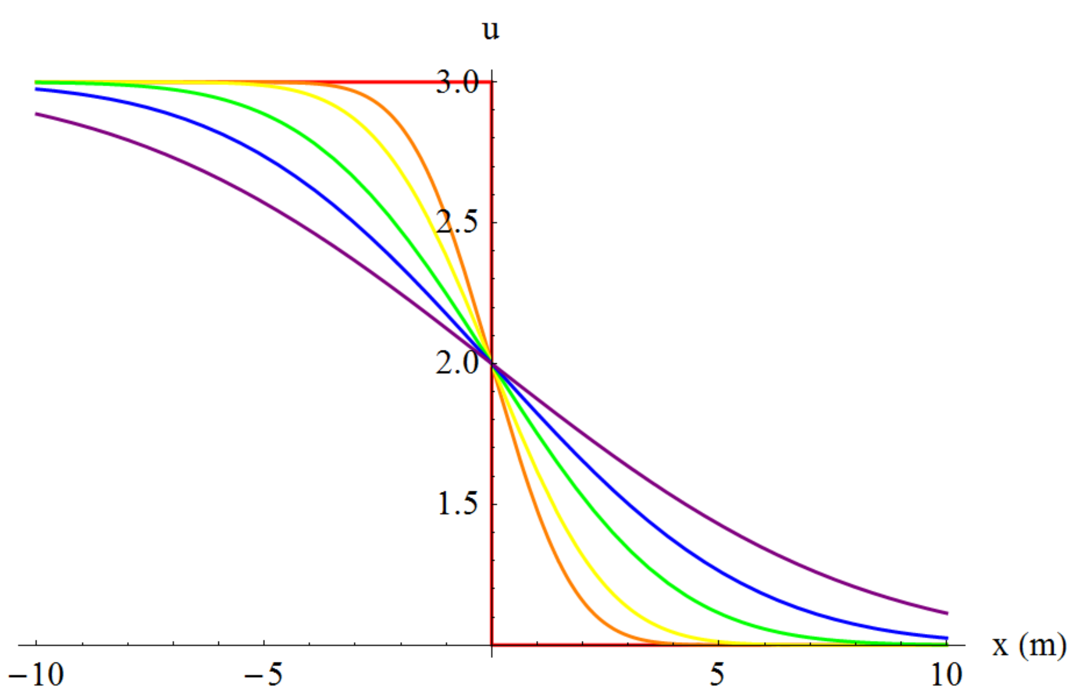

Figure 1: Plot of the solution $u(x,t)$ with $k = 1$ m^2/s for six different times: $t = 0$ s (red), $t = 1$ s (orange), $t = 2$ s (yellow), $t = 5$ s (green), $t = 10$ s (blue), and $t = 20$ s (purple).

Solution by Exploitation of the Invariance Properties

The initial condition,

$$\phi(x) = \begin{cases} 3 & x < 0 \\ 1 & x > 0 \end{cases},$$

can be written in terms of the signum function, which is defined as

$$\operatorname{sgn} x = \begin{cases} 1 & x > 0 \\ -1 & x < 0 \end{cases},$$

as

$$\phi(x) = 2 - \operatorname{sgn} x.$$

Since any linear combination of solutions is a solution to the diffusion equation, we can solve (1) if we can find the solutions with 2 and $\operatorname{sgn} x$ as initial conditions separately. The equations we have to solve are

$$v_t = kv_{xx}, \quad v(x,0) = 2 \tag{6}$$

$$w_t = kw_{xx}, \quad w(x,0) = \operatorname{sgn} x. \tag{7}$$

Once we have v and w, the solution will be $u(x,t) = v(x,t) - w(x,t)$. Right away we can solve (6) since 2 is a constant: $v(x,t) = 2$. To solve (7) we will use the similarity method. Since the signum function is dimensionless, w is dimensionless as well. Hence, the variables x (meters), t (seconds), and k (meters^2/second) have to appear in the solution as a dimensionless combination. The combination variable η is thus

$$\frac{x^2}{kt}$$

or any convenient multiple or power thereof. Choose

$$\eta = \frac{x}{\sqrt{4kt}}$$

to make the process of obtaining the final answer smoother. Hence,

$$w = w(\eta),$$

Write expressions for w_t and w_{xx} in terms of this new variable using the chain rule.

$$\frac{\partial w}{\partial t} = \frac{dw}{d\eta}\frac{\partial \eta}{\partial t} = \frac{dw}{d\eta}\left(-\frac{x}{4\sqrt{kt^3}}\right)$$

$$\frac{\partial w}{\partial x} = \frac{dw}{d\eta}\frac{\partial \eta}{\partial x} = \frac{dw}{d\eta}\left(\frac{1}{\sqrt{4kt}}\right)$$

$$\frac{\partial^2 w}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial w}{\partial x}\right) = \frac{\partial \eta}{\partial x}\frac{\partial}{\partial \eta}\left(\frac{\partial w}{\partial x}\right) = \frac{d^2 w}{d\eta^2}\left(\frac{1}{4kt}\right)$$

Substituting these expressions into (7) gives

$$\frac{dw}{d\eta}\left(-\frac{x}{4\sqrt{kt^3}}\right) = k\frac{d^2 w}{d\eta^2}\left(\frac{1}{4kt}\right).$$

Cancel common terms and bring all terms to one side.

$$w'' + \frac{x}{\sqrt{kt}} w' = 0$$

Use the combination variable η.

$$w'' + 2\eta w' = 0$$

Make the substitution $r = w' = dw/d\eta$.

$$\frac{dr}{d\eta} + 2\eta r = 0$$

This is a first-order differential equation that we can solve by separation of variables.

$$\frac{dr}{r} = -2\eta\, d\eta$$

Integrate both sides.

$$\ln |r| = -\eta^2 + D$$

Exponentiate both sides.

$$|r| = e^{-\eta^2 + D}$$

$$r = \pm e^D e^{-\eta^2}$$

$$r = D_1 e^{-\eta^2}$$

Now that we have r, we can get w by integrating the result.

$$w(\eta) = \int_0^\eta r\, ds + D_2 = \int_0^\eta D_1 e^{-s^2}\, ds + D_2$$

The lower limit, 0, is arbitrary; choosing a different value leads to a different value for D_2. To evaluate the constants of integration, we look to the initial condition, $w(x, 0) = \operatorname{sgn} x$. For $x > 0$, as $t \to 0$, $\eta \to +\infty$. Conversely, for $x < 0$, as $t \to 0$, $\eta \to -\infty$. Thus,

For $x > 0$: $$1 = D_1 \int_0^\infty e^{-s^2}\, ds + D_2$$

For $x < 0$: $$-1 = D_1 \int_0^{-\infty} e^{-s^2}\, ds + D_2$$

The system of equations to solve is

$$1 = D_1 \cdot \frac{\sqrt{\pi}}{2} + D_2$$

$$-1 = -D_1 \cdot \frac{\sqrt{\pi}}{2} + D_2.$$

Solving it yields

$$D_1 = \frac{2}{\sqrt{\pi}} \quad \text{and} \quad D_2 = 0.$$

The solution is thus

$$w(\eta) = \frac{2}{\sqrt{\pi}} \int_0^\eta e^{-s^2}\, ds = \operatorname{erf}(\eta).$$

In terms of the original variables it is

$$w(x,t) = \operatorname{erf}\left(\frac{x}{\sqrt{4kt}}\right).$$

Therefore,

$$u(x,t) = 2 - \operatorname{erf}\left(\frac{x}{\sqrt{4kt}}\right),$$

which is the same result as before.