Exercise 3

Use (8) to solve the diffusion equation if $\phi(x) = e^{3x}$. (You may also use Exercises 6 and 7 below.)

Solution

Equation (8) states the solution to the diffusion equation,

$$u_t = ku_{xx},$$

with the initial condition, $u(x, 0) = \phi(x)$.

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-(x-y)^2/4kt} \phi(y)\, dy \tag{8}$$

Substitute $\phi(x) = e^{3x}$ into the formula.

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{\frac{-(x-y)^2}{4kt}} e^{3y}\, dy$$

Combine the exponentials into one.

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{\frac{-(x-y)^2}{4kt} + 3y}\, dy$$

The exponent E becomes the following.

$$\begin{aligned}
E &= -\frac{(x-y)^2}{4kt} + 3y \\
&= \frac{-x^2 + 2xy - y^2 + 12kty}{4kt} \\
&= \frac{-x^2 - y^2 + (2x + 12kt)y - (x + 6kt)^2 + (x + 6kt)^2}{4kt} \\
&= \frac{-y^2 + 2(x + 6kt)y - (x + 6kt)^2 - x^2 + (x + 6kt)^2}{4kt} \\
&= \frac{-[y - (x + 6kt)]^2 - \cancel{x^2} + \cancel{x^2} + 12ktx + 36k^2t^2}{4kt} \\
&= \frac{12kt(x + 3kt) - (y - x + 6kt)^2}{4kt} \\
&= 3(x + 3kt) - \frac{(y - x - 6kt)^2}{4kt}
\end{aligned}$$

So we have for $u(x,t)$:

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}} e^{3(x+3kt)} \int_{-\infty}^{\infty} e^{-\frac{(y-x-6kt)^2}{4kt}}\, dy.$$

Make the following substitution to solve the integral.

$$p = \frac{y - x - 6kt}{\sqrt{4kt}} \quad \rightarrow \quad p^2 = \frac{(y - x - 6kt)^2}{4kt}$$

$$dp = \frac{dy}{\sqrt{4kt}}$$

The integral becomes

$$u(x,t) = \frac{1}{\sqrt{\pi}} e^{3(x+3kt)} \int_{-\infty}^{\infty} e^{-p^2}\, dp,$$

and it evaluates to $\sqrt{\pi}$.

$$u(x,t) = \frac{1}{\sqrt{\pi}} e^{3(x+3kt)} \cdot \sqrt{\pi}$$

Therefore,

$$u(x,t) = e^{3(x+3kt)}.$$

We can check that this is the solution to the problem.

$$\begin{aligned} u_t &= e^{3(x+3kt)} \cdot 9k \\ u_x &= e^{3(x+3kt)} \cdot 3 \\ u_{xx} &= e^{3(x+3kt)} \cdot 3 \cdot 3 \end{aligned}$$

Thus, $u_t = ku_{xx}$, and this is the correct solution.

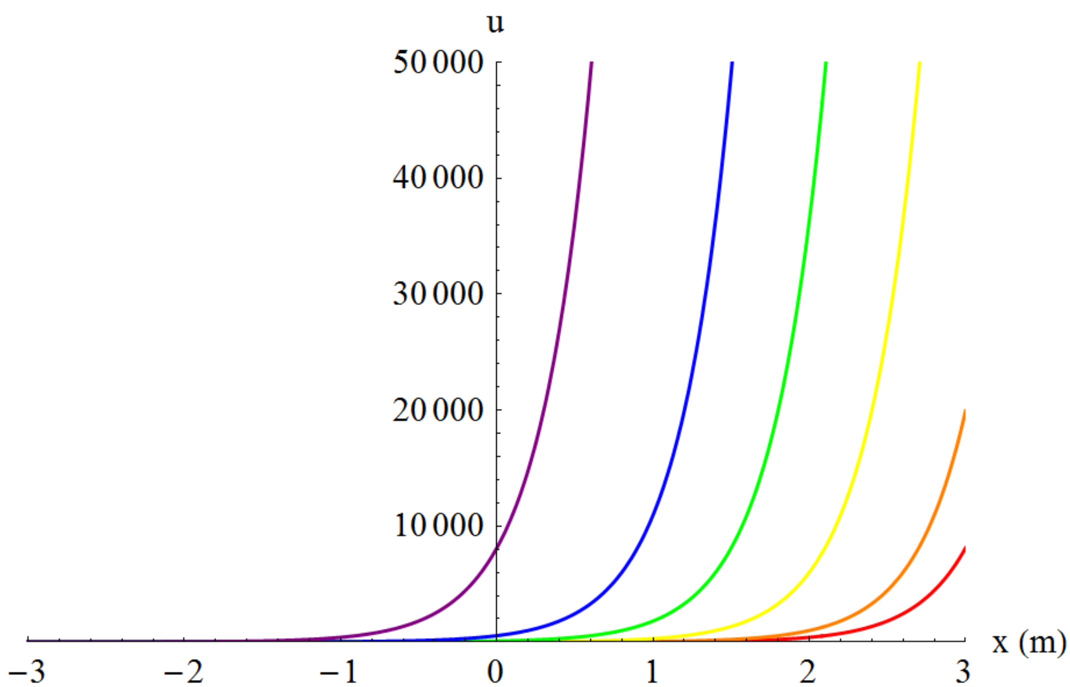

Figure 1: Plot of the solution $u(x,t)$ with $k = 1$ m^2/s for six different times: $t = 0$ s (red), $t = 0.1$ s (orange), $t = 0.3$ s (yellow), $t = 0.5$ s (green), $t = 0.7$ s (blue), and $t = 1$ s (purple).