Exercise 9

Solve the diffusion equation $u_t = ku_{xx}$ with the initial condition $u(x, 0) = x^2$ by the following special method. First show that u_{xxx} satisfies the diffusion equation with *zero* initial condition. Therefore, by uniqueness, $u_{xxx} \equiv 0$. Integrating this result thrice, obtain $u(x, t) = A(t)x^2 + B(t)x + C(t)$. Finally, it's easy to solve for A, B, and C by plugging into the original problem.

Solution

Start by differentiating both sides of the diffusion equation three times with respect to x.

$$(u_t)_{xxx} = (ku_{xx})_{xxx}$$

k can be pulled out of the derivative on the right, and we can change the order of the derivatives to whatever we like.

$$(u_{xxx})_t = k(u_{xxx})_{xx}$$

This implies that u_{xxx} is a solution to the diffusion equation. To find out the initial condition that goes with it, we differentiate both sides of $u(x, 0) = x^2$ three times with respect to x.

$$u(x, 0) = x^2 \quad \rightarrow \quad u_x(x, 0) = 2x \quad \rightarrow \quad u_{xx}(x, 0) = 2 \quad \rightarrow \quad u_{xxx}(x, 0) = 0$$

Thus, u_{xxx} satisfies the diffusion equation with zero initial condition. Since $u_{xxx} = 0$ is a solution to this initial value problem, it is the one and only solution because of uniqueness. We can determine u by integrating both sides partially three times with respect to x.

$$u_{xxx} = 0$$

Integrate both sides with respect to x once.

$$u_{xx} = a(t),$$

where a is an arbitrary function of t. Integrate both sides again with respect to x.

$$u_x = a(t)x + B(t),$$

where B is another arbitrary function of t. Integrate both sides for the last time with respect to x.

$$u(x, t) = \frac{1}{2}a(t)x^2 + B(t)x + C(t),$$

where C is another arbitrary function of t. To get rid of the constant in front of a, let $A(t) = (1/2)a(t)$. This gives us the general formula for $u(x, t)$.

$$u(x, t) = A(t)x^2 + B(t)x + C(t)$$

Now we plug this form into the diffusion equation and its initial condition in order to determine $A(t)$, $B(t)$, and $C(t)$.

$$\begin{aligned} u_t &= A'(t)x^2 + B'(t)x + C'(t) \\ u_x &= 2A(t)x + B(t) \\ u_{xx} &= 2A(t) \end{aligned}$$

Hence,

$$\begin{aligned} u_t &= k u_{xx} \\ A'(t)x^2 + B'(t)x + C'(t) &= 2kA(t). \end{aligned}$$

Matching coefficients on both sides gives us equations we can use to solve for the arbitrary functions.

$$\begin{aligned} A'(t) &= 0 && \rightarrow && A(t) = A_0 \\ B'(t) &= 0 && \rightarrow && B(t) = B_0 \\ C'(t) &= 2kA(t) && \rightarrow && C(t) = 2kA_0 t + C_0 \end{aligned}$$

To determine these arbitrary constants of integration, we have to use the initial condition $u(x, 0) = x^2$.

$$u(x, 0) = A(0)x^2 + B(0)x + C(0) = x^2$$

Matching these coefficients tells us the following.

$$\begin{aligned} A(0) &= A_0 = 1 \\ B(0) &= B_0 = 0 \\ C(0) &= C_0 = 0 \end{aligned}$$

Consequently, the arbitrary functions are determined to be

$$\begin{aligned} A(t) &= 1 \\ B(t) &= 0 \\ C(t) &= 2kt. \end{aligned}$$

Therefore,

$$u(x, t) = x^2 + 2kt.$$

We can check that this is the solution to the diffusion equation.

$$\begin{aligned} u_t &= 2k \\ u_x &= 2x \\ u_{xx} &= 2 \end{aligned}$$

$u_t = k u_{xx}$, so this is the correct solution.

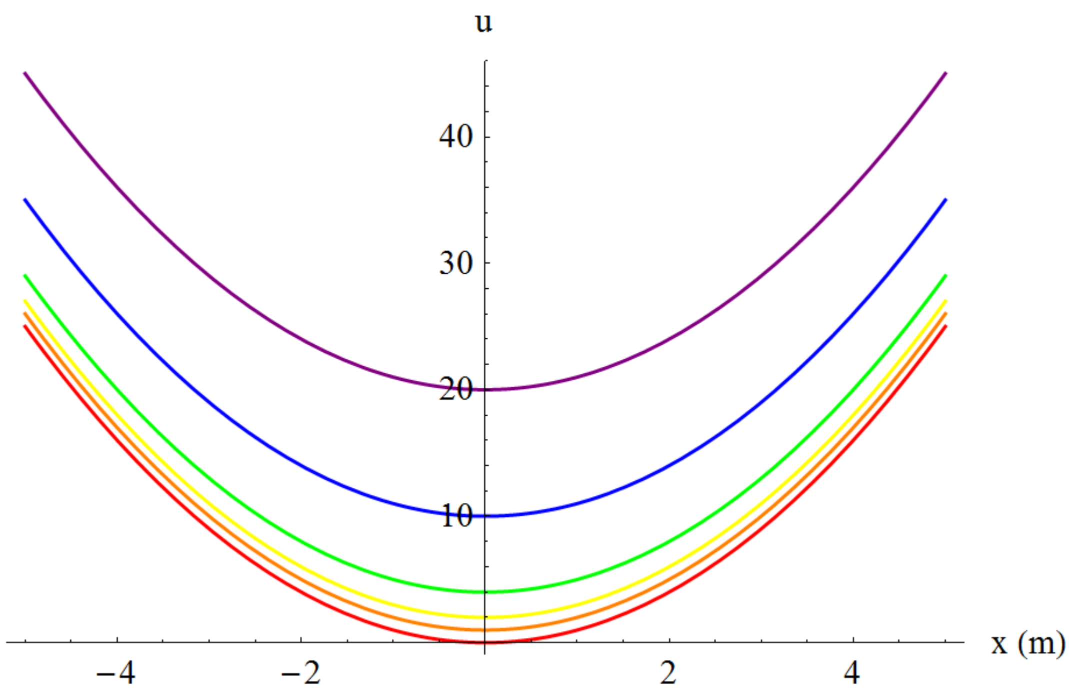

Figure 1: Plot of the solution $u(x,t)$ with $k = 1$ m$^2/s$ for six different times: $t = 0$ s (red), $t = 0.5$ s (orange), $t = 1$ s (yellow), $t = 2$ s (green), $t = 5$ s (blue), and $t = 10$ s (purple).