Exercise 13

Prove from first principles that $Q(x,t)$ *must* have the form (4), as follows.

(a) Assuming uniqueness show that $Q(x,t) = Q(\sqrt{a}x, at)$. This identity is valid for all $a > 0$, all $t > 0$, and all x.

(b) Choose $a = 1/(4kt)$.

Solution

The goal here is to show that

$$Q(x,t) = Q\left(\frac{x}{\sqrt{4kt}}\right)$$

by following steps (a) and (b).

Part (a)

Show that $Q(y = \sqrt{a}x, z = at)$ is a solution of the diffusion equation, $Q_t = kQ_{xx}$. Use the chain rule to write the derivatives of Q in terms of the new variables.

$$\begin{aligned}
\frac{\partial Q}{\partial t} &= \frac{\partial Q}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial Q}{\partial z}\frac{\partial z}{\partial t} = Q_y \cdot 0 + Q_z \cdot a = aQ_z \\
\frac{\partial Q}{\partial x} &= \frac{\partial Q}{\partial y}\frac{\partial y}{\partial x} + \frac{\partial Q}{\partial z}\frac{\partial z}{\partial x} = Q_y \cdot \sqrt{a} + Q_z \cdot 0 = \sqrt{a}Q_y \\
\frac{\partial^2 Q}{\partial x^2} &= \frac{\partial}{\partial x}\left(\frac{\partial Q}{\partial x}\right) = \left(\frac{\partial y}{\partial x}\frac{\partial}{\partial y} + \frac{\partial z}{\partial x}\frac{\partial}{\partial z}\right)(\sqrt{a}Q_y) = \left(\sqrt{a}\cdot\frac{\partial}{\partial y} + 0\cdot\frac{\partial}{\partial z}\right)(\sqrt{a}Q_y) = aQ_{yy}
\end{aligned}$$

Thus, if we substitute the expressions for Q_t and Q_{xx} into the diffusion equation, then we get

$$aQ_z = kaQ_{yy}.$$

After cancelling a we see that

$$Q_z = kQ_{yy},$$

which means that $Q(y = \sqrt{a}x, z = at)$ satisfies the diffusion equation. Because the solution to the diffusion equation is unique, $Q(x,t)$ and $Q(\sqrt{a}x, at)$ must be one and the same.

$$Q(x,t) = Q(\sqrt{a}x, at)$$

Part (b)

If we choose

$$a = \frac{1}{4kt},$$

then we have

$$Q(x,t) = Q\left(\sqrt{\frac{1}{4kt}}x, \frac{1}{4kt}t\right) = Q\left(\frac{x}{\sqrt{4kt}}, \frac{1}{4k}\right).$$

The second argument of Q is only a constant, so we exclude it. Therefore,

$$Q(x,t) = Q\left(\frac{x}{\sqrt{4kt}}\right).$$