Exercise 15

Prove the uniqueness of the diffusion problem with Neumann boundary conditions:

$$u_t - ku_{xx} = f(x,t) \quad \text{for } 0 < x < l,\ t > 0 \quad u(x,0) = \phi(x)$$

$$u_x(0,t) = g(t) \quad u_x(l,t) = h(t)$$

by the energy method.

Solution

What we do here is assume there are two solutions that satisfy this problem and then show that they have to be one and the same. Let $u(x,t)$ and $v(x,t)$ satisfy the diffusion equation and all the conditions.

$$\begin{aligned} u_t &= ku_{xx} & u(x,0) &= \phi(x) & v(x,0) &= \phi(x) \\ v_t &= kv_{xx} & u_x(0,t) &= g(t) & v_x(0,t) &= g(t) \\ t > 0,\ 0 < x &< l & u_x(l,t) &= h(t) & v_x(l,t) &= h(t) \end{aligned}$$

Consider the difference between the two diffusion equations.

$$u_t - v_t = ku_{xx} - kv_{xx}$$

Factor out k and the operators.

$$(u-v)_t = k(u-v)_{xx}$$

Now make the substitution, $w(x,t) = u(x,t) - v(x,t)$.

$$\begin{aligned} w_t &= kw_{xx} & w(x,0) &= u(x,0) - v(x,0) = 0 \\ t > 0,\ 0 < x &< l & w_x(0,t) &= u_x(0,t) - v_x(0,t) = 0 \\ & & w_x(l,t) &= u_x(l,t) - v_x(l,t) = 0 \end{aligned}$$

This shows that w satisfies the diffusion equation with zero initial and boundary conditions. Now proceed with the energy method to prove the uniqueness. Start by multiplying both sides of this diffusion equation by w.

$$ww_t = kww_{xx}$$

Rewrite both sides as follows.

$$\frac{1}{2}(w^2)_t = k[(w_x w)_x - w_x^2]$$

Integrate both sides with respect to x over the domain, $0 < x < l$.

$$\int_0^l \frac{1}{2}(w^2)_t\, dx = \int_0^l k[(w_x w)_x - w_x^2]\, dx$$

Bring the constant and derivative out in front of the integral on the left and distribute the integral on the right.

$$\frac{1}{2}\frac{d}{dt}\int_0^l w^2\, dx = k\left[\int_0^l (w_x w)_x\, dx - \int_0^l w_x^2\, dx\right]$$

Multiply both sides by 2 and integrate the first integral on the right.

$$\frac{d}{dt}\int_0^l w^2\,dx = 2k\left[(w_x w)\big|_0^l - \int_0^l w_x^2\,dx\right]$$

Because of the zero boundary conditions, the first integral evaluates to 0.

$$\frac{d}{dt}\int_0^l w^2\,dx = 2k\left[\underbrace{w_x(l,t)}_{=0}w(l,t) - \underbrace{w_x(0,t)}_{=0}w(0,t) - \int_0^l w_x^2\,dx\right]$$

$$\frac{d}{dt}\int_0^l w^2\,dx = -2k\left[\int_0^l w_x^2\,dx\right]$$

Because w_x^2 is strictly positive, the integral of it is as well. And since a minus sign is present, this differential equation indicates that the quantity,

$$\int_0^l w^2\,dx,$$

decreases over time. Hence, it's highest value must occur initially.

$$\int_0^l [w(x,t)]^2\,dx \le \int_0^l [w(x,0)]^2\,dx$$

But $w(x,0) = 0$, so

$$\int_0^l [w(x,t)]^2\,dx \le 0.$$

As stated previously, the integral cannot be negative because w^2 is always greater than or equal to 0, which means that

$$\int_0^l [w(x,t)]^2\,dx = 0.$$

By the first vanishing theorem, $[w(x,t)]^2 = 0$, so $w(x,t) = 0$. Now we plug back in $w(x,t) = u(x,t) - v(x,t)$ to get $u(x,t) - v(x,t) = 0$. Therefore, $u(x,t) = v(x,t)$, and the solution to the diffusion equation with Neumann boundary conditions is unique.