Exercise 16

Solve the diffusion equation with constant dissipation:

$$u_t - ku_{xx} + bu = 0 \qquad \text{for } -\infty < x < \infty \qquad \text{with } u(x,0) = \phi(x),$$

where $b > 0$ is a constant. (*Hint:* Make the change of variables $u(x,t) = e^{-bt}v(x,t)$.)

Solution

Solution by the Change of Variables

Following the hint, make the change of variables,

$$u(x,t) = e^{-bt}v(x,t). \tag{1}$$

Write expressions for the derivatives of u in terms of this new variable.

$$\begin{aligned} u_t &= -be^{-bt}v + e^{-bt}v_t \\ u_x &= e^{-bt}v_x \\ u_{xx} &= e^{-bt}v_{xx} \end{aligned}$$

Now substitute these into the PDE.

$$u_t - ku_{xx} + bu = (-\cancel{be^{-bt}v} + e^{-bt}v_t) - k(e^{-bt}v_{xx}) + \cancel{be^{-bt}v} = 0$$

After cancelling the common terms, what remains is

$$e^{-bt}v_t - ke^{-bt}v_{xx} = 0.$$

Multiply both sides by e^{bt} and bring the second term to the right.

$$v_t = kv_{xx}$$

This is the diffusion equation. Solving (1) for v gives $v(x,t) = e^{bt}u(x,t)$, so the initial condition is $v(x,0) = e^0u(x,0) = \phi(x)$. Its solution is given as

$$v(x,t) = \frac{1}{\sqrt{4\pi kt}}\int_{-\infty}^{\infty} e^{-\frac{(x-s)^2}{4kt}}\phi(s)\,ds.$$

Change back to the original variable, $u(x,t)$.

$$e^{bt}u(x,t) = \frac{1}{\sqrt{4\pi kt}}\int_{-\infty}^{\infty} e^{-\frac{(x-s)^2}{4kt}}\phi(s)\,ds$$

Therefore,

$$u(x,t) = \frac{e^{-bt}}{\sqrt{4\pi kt}}\int_{-\infty}^{\infty} e^{-\frac{(x-s)^2}{4kt}}\phi(s)\,ds.$$

Solution by the Integrating Factor

Because the PDE is first-order in the derivative with respect to time, we can multiply the PDE by an integrating factor as we would if it were an ODE.

$$u_t + bu - ku_{xx} = 0$$

The integrating factor is

$$I = e^{\int^t b\,ds} = e^{bt}.$$

Now multiply both sides of the PDE by it.

$$e^{bt}u_t + be^{bt}u - ke^{bt}u_{xx} = 0$$

The first two terms on the left side can be written as the time derivative of the product, $e^{bt}u$. Also, e^{bt} can be brought inside the second partial derivative with respect to x because t is considered to be a constant.

$$\frac{\partial}{\partial t}(e^{bt}u) - k\frac{\partial^2}{\partial x^2}(e^{bt}u) = 0$$

Bring the second term to the right.

$$\frac{\partial}{\partial t}(e^{bt}u) = k\frac{\partial^2}{\partial x^2}(e^{bt}u)$$

$e^{bt}u$ satisfies the diffusion equation, and its solution is thus

$$e^{bt}u(x,t) = \frac{1}{\sqrt{4\pi kt}}\int_{-\infty}^{\infty} e^{-\frac{(x-s)^2}{4kt}}\,\phi(s)\,ds.$$

Therefore,

$$u(x,t) = \frac{e^{-bt}}{\sqrt{4\pi kt}}\int_{-\infty}^{\infty} e^{-\frac{(x-s)^2}{4kt}}\,\phi(s)\,ds.$$