Exercise 17

Solve the diffusion equation with variable dissipation:

\[ u_t - ku_{xx} + bt^2 u = 0 \quad \text{for} \quad -\infty < x < \infty \quad \text{with} \ u(x,0) = \phi(x), \]

where \( b > 0 \) is a constant. (\textit{Hint:} The solutions of the ODE \( w_t + bt^2 w = 0 \) are \( Ce^{-bt^3/3} \). So make the change of variables \( u(x,t) = e^{-bt^3/3}v(x,t) \) and derive an equation for \( v \).)

Solution

Solution by the Change of Variables

Following the hint, make the change of variables,

\[ u(x,t) = e^{-bt^3/3}v(x,t). \]  

(1)

Write expressions for the derivatives of \( u \) in terms of this new variable.

\[
\begin{align*}
\frac{\partial u}{\partial t} &= -bt^2 e^{-bt^3/3}v + e^{-bt^3/3}v_t \\
\frac{\partial u}{\partial x} &= e^{-bt^3/3}v_x \\
\frac{\partial^2 u}{\partial x^2} &= e^{-bt^3/3}v_{xx}
\end{align*}
\]

Now substitute these into the PDE.

\[
\frac{\partial u}{\partial t} - k\frac{\partial^2 u}{\partial x^2} + bt^2 u = -bt^2 e^{-bt^3/3}v + e^{-bt^3/3}v_t - k(e^{-bt^3/3}v_{xx}) + bt^2 e^{-bt^3/3}v
\]

After cancelling the common terms, what remains is

\[ e^{-bt^3/3}v_t - ke^{-bt^3/3}v_{xx} = 0. \]

Multiply both sides by \( e^{bt^3/3} \) and bring the second term to the right.

\[ v_t = kv_{xx}. \]

This is the diffusion equation. Solving (1) for \( v \) gives \( v(x,t) = e^{bt^3/3}u(x,t) \), so the initial condition is \( v(x,0) = e^0 u(x,0) = \phi(x) \). Its solution is given as

\[ v(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-\frac{(x-s)^2}{4kt}} \phi(s) \, ds. \]

Change back to the original variable, \( u(x,t) \).

\[
\begin{align*}
e^{bt^3/3}u(x,t) &= \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-\frac{(x-s)^2}{4kt}} \phi(s) \, ds \\
\end{align*}
\]

Therefore,

\[ u(x,t) = e^{-bt^3/3} \int_{-\infty}^{\infty} e^{-\frac{(x-s)^2}{4kt}} \phi(s) \, ds. \]
Solution by the Integrating Factor

Because the PDE is first-order in the derivative with respect to time, we can multiply the PDE by an integrating factor as we would if it were an ODE.

\[ u_t + bt^2 u - ku_{xx} = 0 \]

The integrating factor is

\[ I = e^{\int bt^2 dt} = e^{bt^3/3}. \]

Now multiply both sides of the PDE by it.

\[ e^{bt^3/3} u_t + bt^2 e^{bt^3/3} u - k e^{bt^3/3} u_{xx} = 0 \]

The first two terms on the left side can be written as the time derivative of the product, \( e^{bt^3/3} u \). Also, \( e^{bt^3/3} \) can be brought inside the second partial derivative with respect to \( x \) because \( t \) is considered to be a constant.

\[ \frac{\partial}{\partial t} (e^{bt^3/3} u) - k \frac{\partial^2}{\partial x^2} (e^{bt^3/3} u) = 0 \]

Bring the second term to the right.

\[ \frac{\partial}{\partial t} (e^{bt^3/3} u) = k \frac{\partial^2}{\partial x^2} (e^{bt^3/3} u) \]

\( e^{bt^3/3} u \) satisfies the diffusion equation, and its solution is thus

\[ e^{bt^3/3} u(x, t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-\frac{(x-s)^2}{4kt}} \phi(s) ds. \]

Therefore,

\[ u(x, t) = \frac{e^{-bt^3/3}}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-\frac{(x-s)^2}{4kt}} \phi(s) ds. \]