Exercise 18

Solve the heat equation with convection:

$$u_t - ku_{xx} + Vu_x = 0 \qquad \text{for } -\infty < x < \infty \qquad \text{with } u(x,0) = \phi(x),$$

where V is a constant. (*Hint:* Go to a moving frame of reference by substituting $y = x - Vt$.)

Solution

Following the hint, make the change of variables $y = x - Vt$ as well as $z = t$. Use the chain rule to write expressions for u in terms of these new variables.

$$\begin{aligned} u_t &= \frac{\partial u}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial t} = u_y(-V) + u_z \cdot 1 = -Vu_y + u_z \\ u_x &= \frac{\partial u}{\partial y}\frac{\partial y}{\partial x} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial x} = u_y \cdot 1 + u_z \cdot 0 = u_y \\ u_{xx} &= \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \left(\frac{\partial y}{\partial x}\frac{\partial}{\partial y} + \frac{\partial z}{\partial x}\frac{\partial}{\partial z}\right)\left(\frac{\partial u}{\partial x}\right) = \left(1 \cdot \frac{\partial}{\partial y} + 0 \cdot \frac{\partial}{\partial z}\right)(u_y) = u_{yy} \end{aligned}$$

Now substitute the terms in the PDE for these new expressions.

$$-Vu_y + u_z - ku_{yy} + Vu_y = 0$$

We find that Vu_y cancels, leaving us with

$$u_z - ku_{yy} = 0.$$

Bring the second term to the right side.

$$u_z = ku_{yy}$$

u satisfies the diffusion equation with these new variables. When $z = t = 0$, $y = x$, so the initial condition doesn't change, $u(y,0) = \phi(y)$. The solution is given as

$$u(y,z) = \frac{1}{\sqrt{4\pi kz}} \int_{-\infty}^{\infty} e^{-\frac{(y-s)^2}{4kz}} \phi(s)\, ds.$$

Now change back to the original variables, x and t, by plugging in $y = x - Vt$ and $z = t$. The final answer is as follows.

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-\frac{(x-Vt-s)^2}{4kt}} \phi(s)\, ds$$