Exercise 18

Solve the heat equation with convection:

\[ u_t - ku_{xx} + Vu_x = 0 \quad \text{for } -\infty < x < \infty \quad \text{with } u(x,0) = \phi(x), \]

where \( V \) is a constant. (Hint: Go to a moving frame of reference by substituting \( y = x - Vt \).

Solution

Following the hint, make the change of variables \( y = x - Vt \) as well as \( z = t \). Use the chain rule to write expressions for \( u \) in terms of these new variables.

\[
\begin{align*}
    u_t &= \frac{\partial u}{\partial y} \frac{\partial y}{\partial t} + \frac{\partial u}{\partial z} \frac{\partial z}{\partial t} = u_y(-V) + u_z \cdot 1 = -Vu_y + u_z \\
    u_x &= \frac{\partial u}{\partial y} \frac{\partial y}{\partial x} + \frac{\partial u}{\partial z} \frac{\partial z}{\partial x} = u_y \cdot 1 + u_z \cdot 0 = u_y \\
    u_{xx} &= \frac{\partial}{\partial x} \left( \frac{\partial u}{\partial x} \right) = \left( \frac{\partial y}{\partial x} \frac{\partial}{\partial y} + \frac{\partial z}{\partial x} \frac{\partial}{\partial z} \right) \left( \frac{\partial u}{\partial x} \right) = \left( 1 \cdot \frac{\partial}{\partial y} + 0 \cdot \frac{\partial}{\partial z} \right) (u_y) = u_{yy}
\end{align*}
\]

Now substitute the terms in the PDE for these new expressions.

\[-Vu_y + u_z - ku_{yy} + Vu_y = 0\]

We find that \( Vu_y \) cancels, leaving us with

\[ u_z - ku_{yy} = 0. \]

Bring the second term to the right side.

\[ u_z = ku_{yy} \]

\( u \) satisfies the diffusion equation with these new variables. When \( z = t = 0, y = x \), so the initial condition doesn’t change, \( u(y,0) = \phi(y) \). The solution is given as

\[ u(y,z) = \frac{1}{\sqrt{4\pi kz}} \int_{-\infty}^{\infty} e^{-\frac{(y-s)^2}{4kz}} \phi(s) \, ds. \]

Now change back to the original variables, \( x \) and \( t \), by plugging in \( y = x - Vt \) and \( z = t \). The final answer is as follows.

\[ u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-\frac{(x-Vt-s)^2}{4kt}} \phi(s) \, ds \]