Exercise 19

(a) Show that $S_2(x, y, t) = S(x, t)S(y, t)$ satisfies the diffusion equation $S_t = k(S_{xx} + S_{yy})$.

(b) Deduce that $S_2(x, y, t)$ is the source function for two-dimensional diffusions.

Solution

Part (a)

Recall that $S(x, t)$ is the Green's function for the diffusion equation. That is, it satisfies

$$S_t = kS_{xx}, \quad S(x, 0) = \delta(x).$$

Its solution is

$$S(x, t) = \frac{1}{\sqrt{4\pi kt}} e^{-\frac{x^2}{4kt}}.$$

Whether we use x or y, S still has the same form. In other words, for the diffusion equation,

$$S_t = kS_{yy}, \quad S(y, 0) = \delta(y),$$

the solution is

$$S(y, t) = \frac{1}{\sqrt{4\pi kt}} e^{-\frac{y^2}{4kt}}.$$

Now we define

$$S_2(x, y, t) = S(x, t)S(y, t) = \left[\frac{1}{\sqrt{4\pi kt}} e^{-\frac{x^2}{4kt}}\right]\left[\frac{1}{\sqrt{4\pi kt}} e^{-\frac{y^2}{4kt}}\right] = \frac{1}{4\pi kt} e^{\frac{-(x^2+y^2)}{4kt}}.$$

Start taking derivatives of S_2.

$$\frac{\partial S_2}{\partial t} = -\frac{1}{4\pi kt^2} e^{\frac{-(x^2+y^2)}{4kt}} + \frac{1}{4\pi kt} e^{\frac{-(x^2+y^2)}{4kt}} \cdot \left(\frac{x^2+y^2}{4kt^2}\right) = \frac{1}{16\pi k^2 t^3}(x^2 + y^2 - 4kt) e^{\frac{-(x^2+y^2)}{4kt}}$$

$$\frac{\partial S_2}{\partial x} = \frac{1}{4\pi kt} e^{\frac{-(x^2+y^2)}{4kt}} \cdot \left(\frac{-x}{2kt}\right) = -\frac{x}{8\pi k^2 t^2} e^{\frac{-(x^2+y^2)}{4kt}}$$

$$\frac{\partial^2 S_2}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial S_2}{\partial x}\right) = \frac{\partial}{\partial x}\left[-\frac{x}{8\pi k^2 t^2} e^{\frac{-(x^2+y^2)}{4kt}}\right] = \frac{1}{16\pi k^3 t^3}(x^2 - 2kt) e^{\frac{-(x^2+y^2)}{4kt}}$$

$$\frac{\partial S_2}{\partial y} = \frac{1}{4\pi kt} e^{\frac{-(x^2+y^2)}{4kt}} \cdot \left(\frac{-y}{2kt}\right) = -\frac{y}{8\pi k^2 t^2} e^{\frac{-(x^2+y^2)}{4kt}}$$

$$\frac{\partial^2 S_2}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial S_2}{\partial y}\right) = \frac{\partial}{\partial y}\left[-\frac{y}{8\pi k^2 t^2} e^{\frac{-(x^2+y^2)}{4kt}}\right] = \frac{1}{16\pi k^3 t^3}(y^2 - 2kt) e^{\frac{-(x^2+y^2)}{4kt}}$$

Adding S_{xx} and S_{yy} together gives

$$\frac{\partial^2 S_2}{\partial x^2} + \frac{\partial^2 S_2}{\partial y^2} = \frac{1}{16\pi k^3 t^3}(x^2 + y^2 - 4kt) e^{\frac{-(x^2+y^2)}{4kt}}.$$

Therefore, S_2 satisfies the two-dimensional diffusion equation, $S_t = k(S_{xx} + S_{yy})$.

Part (b)

In order to claim that $S_2(x, y, t)$ is the Green's function for the two-dimensional diffusion equation, we have to show that the general solution to the initial value problem,

$$u_t = k(u_{xx} + u_{yy}), \quad u(x, y, 0) = \phi(x, y), \tag{1}$$

is given by

$$u(x, y, t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} S_2(x - r, y - s, t)\phi(r, s)\, dr\, ds. \tag{2}$$

We will take derivatives of this and then plug it into the PDE to show that it is indeed the solution.

$$\begin{aligned}
u_t &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\partial S_2}{\partial t}(x - r, y - s, t)\phi(r, s)\, dr\, ds \\
u_x &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\partial S_2}{\partial x}(x - r, y - s, t)\phi(r, s)\, dr\, ds \\
u_{xx} &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\partial^2 S_2}{\partial x^2}(x - r, y - s, t)\phi(r, s)\, dr\, ds \\
u_y &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\partial S_2}{\partial y}(x - r, y - s, t)\phi(r, s)\, dr\, ds \\
u_{yy} &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\partial^2 S_2}{\partial y^2}(x - r, y - s, t)\phi(r, s)\, dr\, ds
\end{aligned}$$

Substitute these expressions for the terms in the PDE.

$$\begin{aligned}
\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\partial S_2}{\partial t}(x - r, y - s, t)\phi(r, s)\, dr\, ds = k\bigg[&\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\partial^2 S_2}{\partial x^2}(x - r, y - s, t)\phi(r, s)\, dr\, ds \\
&+ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\partial^2 S_2}{\partial y^2}(x - r, y - s, t)\phi(r, s)\, dr\, ds\bigg]
\end{aligned}$$

Combine the two integrals on the right and factor $\phi(r, s)$. Also, bring k inside the double integral.

$$\begin{aligned}
\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\partial S_2}{\partial t}(x - r, y - s, t)\phi(r, s)\, dr\, ds = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} k\bigg[&\frac{\partial^2 S_2}{\partial x^2}(x - r, y - s, t) \\
&+ \frac{\partial^2 S_2}{\partial y^2}(x - r, y - s, t)\bigg]\phi(r, s)\, dr\, ds
\end{aligned}$$

We showed in part (a) that S_2 satisfies the two-dimensional diffusion equation. It turns out that any translate of S_2 also satisfies the diffusion equation as is the case in one dimension. Thus,

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\partial S_2}{\partial t}(x - r, y - s, t)\phi(r, s)\, dr\, ds = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\partial S_2}{\partial t}(x - r, y - s, t)\phi(r, s)\, dr\, ds.$$

Because both sides are equal, (2) is the general solution to (1). In order to get the initial condition $\phi(x, y)$ when $t = 0$, we require that $S_2(x, y, 0) = \delta(x)\delta(y)$ so that $\phi(x, y)$ is "sifted" out of the double integral in (2). This is true, though, since $S_2(x, y, t) = S(x, t)S(y, t)$. $S_2(x, y, 0) = S(x, 0)S(y, 0) = \delta(x)\delta(y)$. Therefore, $S_2(x, y, t)$ is the Green's function to the two-dimensional diffusion equation.