Exercise 4

Solve the diffusion equation if $\phi(x) = e^{-x}$ for $x > 0$ and $\phi(x) = 0$ for $x < 0$.

Solution

Solution by the Similarity Method

We have to solve the initial value problem,

$$u_t = ku_{xx}, \quad u(x,0) = \phi(x). \tag{1}$$

In order to do so, we’ll solve for the Green’s function $G(x,t)$ in the corresponding PDE,

$$G_t = kG_{xx}, \quad G(x,0) = \delta(x), \tag{2}$$

where $\delta(x)$, the Dirac delta function, is defined as

$$\delta(x) = \begin{cases} 1 & x = 0 \\ 0 & x \neq 0 \end{cases}.$$

The reason we’re solving the equation with the delta function is that it has the extremely useful “sifting” property,

$$\int_{-\infty}^{\infty} f(s)\delta(x - s) \, ds = f(x),$$

so the solution to the initial value problem (1) in terms of the Green’s function is

$$u(x,t) = \int_{-\infty}^{\infty} G(x-s,t)\phi(s) \, ds.$$

This can be verified by substituting this form for $u$ into (1). Now we will go about solving (2) for $G(x,t)$ by using the similarity method (also known as the combination of variables method). Because $u$ is a dimensionless quantity (that is, it yields a pure number with no units) the variables $x$, $t$, and $k$ have to appear in the solution in a dimensionless combination. $x$ has units of meters, $t$ has units of seconds, and $k$ has units of meters$^2$/second, so the combination of variables has to be

$$\frac{x^2}{kt}$$

or any convenient multiple or power thereof. Therefore,

$$u = u(\eta), \quad \text{where we choose } \eta = \frac{x}{\sqrt{kt}}.$$

We choose this particular form for $\eta$ so the process of getting the final answer is smoother. We’re trying to solve (2) for $G$, though, and $G$ is not dimensionless; as can be seen from the initial condition, it has the same dimensions as $\delta(x)$. $\delta(x)$ has the inverse dimension of its argument, so $G$ has dimensions of meters$^{-1}$. Thus, $G$ has to be of the form,

$$G(x,t) = \frac{1}{\sqrt{kt}}g\left(\frac{x}{\sqrt{kt}}\right),$$

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where $g$ is an arbitrary function. In order to determine $g$, we have to plug this form into (2) and solve the resulting ODE. To start, write the expressions for $G_t$ and $G_{xx}$.

\[
\frac{\partial G}{\partial t} = -\frac{1}{2\sqrt{kt^3}}g + \frac{1}{\sqrt{kt}} \left(-\frac{x}{2\sqrt{kt^3}}\right) g' \\
\frac{\partial G}{\partial x} = \frac{1}{\sqrt{kt}} \cdot \frac{1}{\sqrt{kt}} g' = \frac{1}{kt} g' \\
\frac{\partial^2 G}{\partial x^2} = \frac{1}{kt} \cdot \frac{1}{\sqrt{kt}} g'' = \frac{1}{\sqrt{k^3t^3}} g''
\]

Substituting these expressions into (2) gives

\[
-\frac{1}{2\sqrt{kt^3}}g - \frac{1}{2\sqrt{kt^3}} \cdot \frac{x}{\sqrt{kt}} g' = \frac{1}{\sqrt{kt^3}} g''.
\]

Cancel common terms and move everything to one side.

\[
g'' + \frac{1}{2} \frac{x}{\sqrt{kt}} g' + \frac{1}{2} g = 0
\]

Use the combination variable $\eta$.

\[
g'' + \frac{\eta}{2} g' + \frac{1}{2} g = 0
\]

The last two terms on the left side can be written as one using the product rule.

\[
g'' + \left(\frac{\eta}{2} g\right)' = 0
\]

Integrate both sides of the equation.

\[
g' + \frac{\eta}{2} g = C_1
\]

This is an inhomogeneous first-order linear differential equation that can be solved with an integrating factor. The integrating factor is

\[
I = e^{\int \frac{\eta}{2} \, d\eta} = e^{\frac{\eta^2}{4}}.
\]

Multiply both sides by $I$.

\[
e^{\frac{\eta^2}{4}} g' + \frac{\eta}{2} e^{\frac{\eta^2}{4}} g = C_1 e^{\frac{\eta^2}{4}}
\]

The two terms on the left side can be written as one using the product rule.

\[
\left(e^{\frac{\eta^2}{4}} g\right)' = C_1 e^{\frac{\eta^2}{4}}
\]

Integrate both sides of the equation a second time.

\[
e^{\frac{\eta^2}{4}} g = \int_1^n C_1 e^{\frac{\eta^2}{4}} \, ds + C_2
\]

Hence, the arbitrary function $g$ is

\[
g(\eta) = e^{-\frac{\eta^2}{4}} \left[C_1 \int_1^n e^{\frac{\eta^2}{4}} \, ds + C_2\right],
\]

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and consequently, the Green’s function is

\[
G = \frac{1}{\sqrt{kt}} g(\eta) = e^{-\frac{x^2}{4kt}} \left[ C_1 \int_0^\eta e^{\frac{s^2}{4kt}} \, ds + C_2 \right].
\]  \tag{3}

The next order of business is to determine the constants of integration, \(C_1\) and \(C_2\). We need to return to the diffusion equation and the initial condition in (2) to figure these out.

\[
G_t = kG_{xx}
\]

Integrate both sides of the equation with respect to \(x\) over the whole line.

\[
\int_{-\infty}^\infty G_t \, dx = \int_{-\infty}^\infty kG_{xx} \, dx
\]

Take out the time derivative from the left side and evaluate the right side.

\[
\frac{d}{dt} \int_{-\infty}^\infty G \, dx = kG_x \bigg|_{-\infty}^\infty
\]

We assume that \(G\) and \(G_x\) tend to 0 as \(x \to \pm \infty\), so we have

\[
\frac{d}{dt} \int_{-\infty}^\infty G \, dx = 0.
\]

This implies that the quantity,

\[
\int_{-\infty}^\infty G \, dx,
\]

remains constant for all time. Initially \(G(x, 0) = \delta(x)\), so

\[
\int_{-\infty}^\infty G(x, t) \, dx = \int_{-\infty}^\infty G(x, 0) \, dx = \int_{-\infty}^\infty \delta(x) \, dx = 1.
\]  \tag{4}

In order for this integral to converge, \(C_1\) has to be 0. In terms of \(x\) and \(t\), (3) becomes

\[
G(x, t) = \frac{C_2}{\sqrt{kt}} e^{-\frac{x^2}{4kt}}.
\]

\(C_2\) can be thought of as a normalization constant that we determine by plugging into (4).

\[
\int_{-\infty}^\infty G(x, t) \, dx = \int_{-\infty}^\infty \frac{C_2}{\sqrt{kt}} e^{-\frac{x^2}{4kt}} \, dx = 1
\]

Make the following substitution to solve the integral.

\[
v = \frac{x}{\sqrt{4kt}}
\]

\[
dv = \frac{dx}{\sqrt{4kt}} \quad \rightarrow \quad 2dv = \frac{dx}{\sqrt{kt}}
\]

The integral becomes

\[
2C_2 \int_{-\infty}^\infty e^{-v^2} \, dv = 1,
\]

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and it evaluates to \( \sqrt{\pi} \).

\[
2C_2 \sqrt{\pi} = 1
\]

Solving for \( C_2 \) yields

\[
C_2 = \frac{1}{\sqrt{4\pi}}.
\]

Therefore, the Green’s function is

\[
G(x, t) = \frac{1}{\sqrt{4\pi kt}} e^{-\frac{x^2}{4kt}},
\]

and the solution to the initial value problem in (1) is

\[
u(x, t) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi kt}} e^{-\frac{(x-s)^2}{4kt}} \phi(s) \, ds.
\]

(5)

\( u(x, t) \) can be interpreted as the convolution of the initial condition with a Gaussian filter. At every point \( x \), \( u(x, t) \) is an averaged, or smoothed, version of the initial condition over an interval of width \( \sqrt{kt} \). As \( t \) increases, the range of the filter grows and \( u(x, t) \) becomes increasingly smooth over \( x \). Any discontinuities or kinks that are present in the initial condition are smoothed out. In this exercise, the initial condition is

\[
\phi(x) = \begin{cases} 
  e^{-x} & x > 0 \\
  0 & x < 0 
\end{cases}.
\]

If we substitute this into the formula in (5), then we get

\[
u(x, t) = \int_{0}^{\infty} \frac{1}{\sqrt{4\pi kt}} e^{-\frac{(x-s)^2}{4kt}} e^{-s} \, ds.
\]

Combine the exponentials into one.

\[
u(x, t) = \int_{0}^{\infty} \frac{1}{\sqrt{4\pi kt}} e^{-\frac{(x-s)^2}{4kt} - s} \, ds.
\]

The exponent \( E \) becomes the following.

\[
E = -\frac{(x-s)^2}{4kt} - s
\]

\[
= \frac{-x^2 - 2xs - s^2 - 4ks}{4kt}
\]

\[
= \frac{-x^2 - s^2 + (2x - 4kt)s - (x - 2kt)^2 + (x - 2kt)^2}{4kt}
\]

\[
= \frac{-s^2 + 2(x - 2kt)s - (x - 2kt)^2 - x^2 + (x - 2kt)^2}{4kt}
\]

\[
= \frac{-[s - (x - 2kt)]^2 - x^2 + 4ktx + 4k^2t^2}{4kt}
\]

\[
= \frac{4kt(kt - x) - (s - x + 2kt)^2}{4kt}
\]

\[
= kt - x - \frac{(s - x + 2kt)^2}{4kt}
\]
So we have for \( u(x,t) \):
\[
u(x,t) = \frac{1}{\sqrt{4\pi kt}} e^{kt-x} \int_{0}^{\infty} e^{-\frac{(s-x+2kt)^2}{4kt}} ds.
\]

Make the following substitution to solve the integral.
\[
p = \frac{s-x+2kt}{\sqrt{4kt}} \quad \rightarrow \quad p^2 = \frac{(s-x+2kt)^2}{4kt}
\]
\[
dp = \frac{ds}{\sqrt{4kt}}
\]

The integral becomes
\[
u(x,t) = \frac{1}{\sqrt{\pi}} e^{kt-x} \int_{-\frac{s-x+2kt}{\sqrt{4kt}}}^{\infty} e^{-p^2} dp.
\]

Split up the integral into two to get desired limits of integration.
\[
u(x,t) = \frac{1}{\sqrt{\pi}} e^{kt-x} \left( \int_{0}^{\infty} e^{-p^2} dp + \int_{0}^{\infty} e^{-p^2} dp \right)
\]

Switch the limits on the first integral and add a minus sign.
\[
u(x,t) = \frac{1}{\sqrt{\pi}} e^{kt-x} \left( - \int_{0}^{\infty} e^{-p^2} dp + \int_{0}^{\infty} e^{-p^2} dp \right)
\]

The error function, \( \text{erf} z \), is defined as
\[
\text{erf} z = \frac{2}{\sqrt{\pi}} \int_{0}^{z} e^{-p^2} dp,
\]
so we can write the first integral in terms of it. Also, the second integral can be evaluated to \( \sqrt{\pi}/2 \).
\[
u(x,t) = \frac{1}{\sqrt{\pi}} e^{kt-x} \left[ - \frac{\sqrt{\pi}}{2} \text{erf} \left( -\frac{x+2kt}{\sqrt{4kt}} \right) + \frac{\sqrt{\pi}}{2} \right]
\]

Therefore,
\[
u(x,t) = \frac{1}{2} e^{kt-x} \left[ 1 - \text{erf} \left( -\frac{x+2kt}{\sqrt{4kt}} \right) \right].
\]
Figure 1: Plot of the solution $u(x, t)$ with $k = 1 \text{ m}^2/\text{s}$ for six different times: $t = 0 \text{ s}$ (red), $t = 0.1 \text{ s}$ (orange), $t = 0.3 \text{ s}$ (yellow), $t = 1 \text{ s}$ (green), $t = 3 \text{ s}$ (blue), and $t = 10 \text{ s}$ (purple).