Exercise 5

Prove properties (a) to (e) of the diffusion equation (1).

Solution

The aim here is to prove the invariance properties of the diffusion equation listed below.

(a) The translate $u(x - y, t)$ of any solution $u(x, t)$ is another solution for any fixed $y$.

(b) Any derivative of a solution is again a solution.

(c) A linear combination of solutions is again a solution.

(d) An integral of solutions is again a solution.

(e) If $u(x, t)$ is a solution, then so is the dilated function $u(\sqrt{a}x, at)$ for any $a > 0$.

Proof of Property (a)

Show that $u(z = x - y, t)$ is a solution of the diffusion equation, $u_t = ku_{xx}$. Use the chain rule to write the derivatives of $u$.

\[
\frac{\partial u}{\partial t} = \frac{\partial u}{\partial z} \frac{\partial z}{\partial t} + \frac{\partial u}{\partial t} \frac{\partial t}{\partial t} = u_z \cdot 0 + u_t \cdot 1 = u_t
\]

\[
\frac{\partial u}{\partial x} = \frac{\partial u}{\partial z} \frac{\partial z}{\partial x} + \frac{\partial u}{\partial t} \frac{\partial t}{\partial x} = u_z \cdot 1 + u_t \cdot 0 = u_z
\]

\[
\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x} \left( \frac{\partial u}{\partial x} \right) = \frac{\partial z}{\partial x} \frac{\partial}{\partial z} (u_z) = 1 \cdot u_{zz} = u_{zz}
\]

Thus, if we substitute the expressions for $u_t$ and $u_{xx}$ into the diffusion equation, then we get

\[u_t = ku_{zz}.
\]

This means that $u(x - y, t)$ satisfies the diffusion equation, and any translate of a solution is also a solution.

Proof of Property (b)

Show that any derivative of $u$ ($u_t$, $u_x$, $u_{xx}$, etc.) satisfies the diffusion equation,

\[u_t = ku_{xx}.
\]

Take any permutation of $n$ derivatives with respect to $x$ and $t$ of both sides.

\[(u_t)_n = (ku_{xx})_n
\]

Bring the constant out in front.

\[(u_t)_n = k(u_{xx})_n
\]

Since it doesn’t matter what order we take the derivatives, we can interchange them to our liking.

\[(u_n)_t = k(u_n)_{xx}
\]
Therefore, \( u_n \) satisfies the diffusion equation, and any derivative of a solution is also a solution.

**Proof of Property (c)**

We have to show that any linear combination of solutions is also a solution to the diffusion equation, \( u_t = ku_{xx} \). Suppose that \( u_1(x,t), u_2(x,t), \ldots, u_n(x,t) \) are solutions to the diffusion equation. Then

\[
\begin{align*}
\frac{\partial u_1}{\partial t} &= k \frac{\partial^2 u_1}{\partial x^2} \\
\frac{\partial u_2}{\partial t} &= k \frac{\partial^2 u_2}{\partial x^2} \\
&\vdots \\
\frac{\partial u_n}{\partial t} &= k \frac{\partial^2 u_n}{\partial x^2}.
\end{align*}
\]

Multiply both sides of the \( i \)th equation by an arbitrary constant \( a_i \).

\[
\begin{align*}
a_1 \frac{\partial u_1}{\partial t} &= k a_1 \frac{\partial^2 u_1}{\partial x^2} \\
a_2 \frac{\partial u_2}{\partial t} &= k a_2 \frac{\partial^2 u_2}{\partial x^2} \\
&\vdots \\
a_n \frac{\partial u_n}{\partial t} &= k a_n \frac{\partial^2 u_n}{\partial x^2}.
\end{align*}
\]

Add all these equations together.

\[
a_1 \frac{\partial u_1}{\partial t} + a_2 \frac{\partial u_2}{\partial t} + \cdots + a_n \frac{\partial u_n}{\partial t} = k \left( a_1 \frac{\partial^2 u_1}{\partial x^2} + a_2 \frac{\partial^2 u_2}{\partial x^2} + \cdots + a_n \frac{\partial^2 u_n}{\partial x^2} \right)
\]

Factor \( k \) from the right side.

\[
\frac{\partial}{\partial t} \left( a_1 u_1 + a_2 u_2 + \cdots + a_n u_n \right) = k \left( \frac{\partial^2}{\partial x^2} (a_1 u_1) + \frac{\partial^2}{\partial x^2} (a_2 u_2) + \cdots + \frac{\partial^2}{\partial x^2} (a_n u_n) \right)
\]

The constants can be put inside the derivatives.

\[
\frac{\partial}{\partial t} \left( a_1 u_1 + a_2 u_2 + \cdots + a_n u_n \right) = k \left( \frac{\partial^2}{\partial x^2} (a_1 u_1 + a_2 u_2 + \cdots + a_n u_n) \right)
\]

The sum of the derivatives is equal to the derivative of the sum, so we can factor out the derivatives from each side.

\[
\frac{\partial}{\partial t} \left( a_1 u_1 + a_2 u_2 + \cdots + a_n u_n \right) = k \frac{\partial^2}{\partial x^2} \left( a_1 u_1 + a_2 u_2 + \cdots + a_n u_n \right)
\]

Therefore, the linear combination of solutions satisfies the diffusion equation and is a solution as well.
Proof of Property (d)

Here we have to show that any integral of a solution to the diffusion equation is also a solution. Suppose then that \( u(x, t) \) is a solution, that is,

\[
  u_t = k u_{xx}.
\]

According to property (a), the translate \( u(x - s, t) \) is a solution as well.

\[
  \frac{\partial u}{\partial t}(x - s, t) = k \frac{\partial^2 u}{\partial x^2}(x - s, t)
\]

Multiply both sides of the equation by an integrable function \( f(s) \).

\[
  \frac{\partial u}{\partial t}(x - s, t) f(s) = k \frac{\partial^2 u}{\partial x^2}(x - s, t) f(s)
\]

Integrate both sides with respect to \( s \) over the whole line.

\[
  \int_{-\infty}^{\infty} \frac{\partial u}{\partial t}(x - s, t) f(s) \, ds = k \int_{-\infty}^{\infty} \frac{\partial^2 u}{\partial x^2}(x - s, t) f(s) \, ds
\]

The constant \( k \) can be pulled in front of the integral.

\[
  \int_{-\infty}^{\infty} \frac{\partial u}{\partial t}(x - s, t) f(s) \, ds = k \int_{-\infty}^{\infty} \frac{\partial^2 u}{\partial x^2}(x - s, t) f(s) \, ds
\]

Since the variable of integration is \( s \), the partial derivatives with respect to \( t \) and \( x \) can be pulled in front of the integrals.

\[
  \frac{\partial}{\partial t} \left[ \int_{-\infty}^{\infty} u(x - s, t) f(s) \, ds \right] = k \frac{\partial^2}{\partial x^2} \left[ \int_{-\infty}^{\infty} u(x - s, t) f(s) \, ds \right]
\]

Therefore, the integral of a solution to the diffusion equation also satisfies the equation and is a solution as well.

Proof of Property (e)

Show that \( u(y = \sqrt{ax}, z = at) \) is a solution of the diffusion equation, \( u_t = k u_{xx} \). Use the chain rule to write the derivatives of \( u \) in terms of the new variables.

\[
  \frac{\partial u}{\partial t} = \frac{\partial u}{\partial y} \frac{\partial y}{\partial t} + \frac{\partial u}{\partial z} \frac{\partial z}{\partial t} = u_y \cdot 0 + u_z \cdot a = au_z
\]

\[
  \frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} \frac{\partial y}{\partial x} + \frac{\partial u}{\partial z} \frac{\partial z}{\partial x} = u_y \cdot \sqrt{a} + u_z \cdot 0 = \sqrt{a} u_y
\]

\[
  \frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x} \left( \frac{\partial u}{\partial x} \right) = \left( \frac{\partial u}{\partial x} \frac{\partial}{\partial y} + \frac{\partial u}{\partial x} \frac{\partial}{\partial z} \right) (\sqrt{a} u_y) = \left( \sqrt{a} \cdot \frac{\partial}{\partial y} + 0 \cdot \frac{\partial}{\partial z} \right) (\sqrt{a} u_y) = au_{yy}
\]

Thus, if we substitute the expressions for \( u_t \) and \( u_{xx} \) into the diffusion equation, then we get

\[
  au_z = k au_{yy}.
\]

After cancelling \( a \) we see that

\[
  au_z = k au_{yy},
\]

which means that \( u(y = \sqrt{ax}, z = at) \) satisfies the diffusion equation and any dilated solution is also a solution.