Exercise 6

Compute $\int_0^\infty e^{-x^2}\,dx$. (*Hint:* This is a function that *cannot* be integrated by formula. So use the following trick. Transform the double integral $\int_0^\infty e^{-x^2}\,dx \cdot \int_0^\infty e^{-y^2}\,dy$ into polar coordinates and you'll end up with a function that can be integrated easily.)

Solution

Following the hint, we will calculate this improper integral (call it I) by considering the indicated double integral and switching to polar coordinates.

$$I = \int_0^\infty e^{-x^2}\,dx$$

Square both sides of the equation.

$$I^2 = \left(\int_0^\infty e^{-x^2}\,dx\right)^2$$

Write one of the integrals on the right with y for the dummy variable.

$$I^2 = \left(\int_0^\infty e^{-x^2}\,dx\right)\left(\int_0^\infty e^{-y^2}\,dy\right)$$

We can combine these two integrals on the right into a double integral.

$$I^2 = \int_0^\infty \int_0^\infty e^{-x^2-y^2}\,dx\,dy$$

Now switch to polar coordinates, $r^2 = x^2 + y^2$ and $dx\,dy = r\,dr\,d\theta$. The limits of integration indicate that we are integrating over the first quadrant, so r will go from 0 to ∞ and θ will go from 0 to $\pi/2$.

$$I^2 = \int_0^{\frac{\pi}{2}} \int_0^\infty e^{-r^2} r\,dr\,d\theta$$

Make a substitution to solve the integral.

$$\begin{aligned} p &= -r^2 \\ dp &= -2r\,dr \quad \rightarrow \quad -\frac{1}{2}\,dp = r\,dr \end{aligned}$$

The double integral becomes

$$I^2 = \int_0^{\frac{\pi}{2}} \int_0^{-\infty} e^p \left(-\frac{1}{2}\right) dp\,d\theta.$$

Bring the constant in front of the integral and carry out the integration for p.

$$I^2 = -\frac{1}{2}\int_0^{\frac{\pi}{2}} e^p\big|_0^{-\infty}\,d\theta$$

Evaluate the limits.

$$I^2 = -\frac{1}{2}\int_0^{\frac{\pi}{2}} (0-1)\,d\theta$$

Bring the constant in front of the integral.

$$I^2 = \frac{1}{2}\int_0^{\frac{\pi}{2}} d\theta$$

Now we carry out the integration for θ.

$$I^2 = \frac{1}{2}\left(\frac{\pi}{2} - 0\right)$$

Thus,

$$I^2 = \frac{\pi}{4}.$$

Lastly, take the square root of both sides to get the desired result.

$$I = \int_0^{\infty} e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}$$