Exercise 7

Use Exercise 6 to show that $\int_{-\infty}^{\infty} e^{-p^2}\, dp = \sqrt{\pi}$. Then substitute $p = x/\sqrt{4kt}$ to show that

$$\int_{-\infty}^{\infty} S(x,t)\, dx = 1.$$

Solution

From Exercise 6 we know that

$$\int_0^{\infty} e^{-p^2}\, dp = \frac{\sqrt{\pi}}{2}.$$

Because e^{-p^2} is an even function, we can extend the lower limit to $-\infty$ so long as we divide the integral by 2.

$$\frac{1}{2}\int_{-\infty}^{\infty} e^{-p^2}\, dp = \frac{\sqrt{\pi}}{2}$$

Multiply both sides of the equation by 2 to get the desired result.

$$\int_{-\infty}^{\infty} e^{-p^2}\, dp = \sqrt{\pi}$$

Now make the substitution,

$$\begin{aligned} p &= \frac{x}{\sqrt{4kt}} \quad \rightarrow \quad p^2 = \frac{x^2}{4kt} \\ dp &= \frac{dx}{\sqrt{4kt}}. \end{aligned}$$

The integral becomes

$$\int_{-\infty}^{\infty} e^{-\frac{x^2}{4kt}} \frac{dx}{\sqrt{4kt}} = \sqrt{\pi}.$$

$S(x,t)$ is defined in the textbook to be the Green's function for the diffusion equation.

$$S(x,t) = \frac{1}{\sqrt{4\pi kt}} e^{-\frac{x^2}{4kt}}$$

We have everything in the integral except for $\sqrt{\pi}$ in the denominator. Substituting $S(x,t)$ gives

$$\int_{-\infty}^{\infty} \sqrt{\pi} S(x,t)\, dx = \sqrt{\pi}.$$

Bring the constant in front of the integral.

$$\sqrt{\pi}\int_{-\infty}^{\infty} S(x,t)\, dx = \sqrt{\pi}$$

Divide both sides by $\sqrt{\pi}$ to get the final result.

$$\int_{-\infty}^{\infty} S(x,t)\, dx = 1$$