Exercise 8

Show that for any fixed $\delta > 0$ (no matter how small),

$$\max_{\delta \leq |x| < \infty} S(x, t) \to 0 \quad \text{as } t \to 0.$$  

[This means that the tail of $S(x, t)$ is “uniformly small”]

Solution

$S(x, t)$ is defined in the textbook to be the Green’s function for the diffusion equation.

$$S(x, t) = \frac{1}{\sqrt{4\pi kt}} e^{-\frac{x^2}{4kt}}$$

The maximum occurs where $x$ is smallest in magnitude because of the negative sign in the exponent. Since $\delta \leq |x| < \infty$, we set $x^2 = \delta^2$.

$$S(\delta, t) = \frac{1}{\sqrt{4\pi kt}} e^{-\frac{\delta^2}{4kt}}$$

Now we have to show that the limit of $S(\delta, t)$ as $t \to 0$ is 0.

$$\lim_{t \to 0} S(\delta, t) = \lim_{t \to 0} \frac{1}{\sqrt{4\pi kt}} e^{-\frac{\delta^2}{4kt}}$$

Bring the constants in front of the limit.

$$\lim_{t \to 0} S(\delta, t) = \frac{1}{\sqrt{4\pi k}} \lim_{t \to 0} \frac{1}{\sqrt{t}} e^{-\frac{\delta^2}{4kt}}$$

Plugging in $t = 0$ yields the indeterminate form 0/0. However, applying L'Hôpital’s rule doesn’t lead to any simplification, so we have to try something else. Proceed by bringing $\sqrt{t}$ to the exponent of the exponential function.

$$\lim_{t \to 0} S(\delta, t) = \frac{1}{\sqrt{4\pi k}} \lim_{t \to 0} e^{-\frac{\delta^2}{4kt} - \ln \sqrt{t}}$$

Factor out the minus sign and bring the limit into the exponent.

$$\lim_{t \to 0} S(\delta, t) = \frac{1}{\sqrt{4\pi k}} e^{-\frac{\delta^2}{4kt} \ln \sqrt{t}}$$

Plugging in $t = 0$ now yields the indeterminate form $\infty - \infty$, so factor out $\delta^2/4kt$ in order to make this a product. Also, change $\sqrt{t}$ to $t$ by bringing a factor of 1/2 in front.

$$\lim_{t \to 0} \left( \frac{\delta^2}{4kt} + \ln \sqrt{t} \right) = \lim_{t \to 0} \frac{\delta^2}{4kt} \left( 1 + \frac{2kt}{\delta^2 \ln t} \right)$$

Since the limit of a product is the product of the limits, we can write this as

$$\lim_{t \to 0} \left( \frac{\delta^2}{4kt} + \ln \sqrt{t} \right) = \left( \lim_{t \to 0} \frac{\delta^2}{4kt} \right) \left( \lim_{t \to 0} \left( 1 + \frac{2kt}{\delta^2 \ln t} \right) \right) .$$
The limit of a sum is the sum of the limits, so
\[
\lim_{t \to 0} \left( \frac{\delta^2}{4kt} + \ln \sqrt{t} \right) = \left( \lim_{t \to 0} \frac{\delta^2}{4kt} \right) \left( \lim_{t \to 0} 1 + \lim_{t \to 0} \frac{2kt}{\delta^2} \ln t \right).
\]

Bring the constants in front of the limits.
\[
\lim_{t \to 0} \left( \frac{\delta^2}{4kt} + \ln \sqrt{t} \right) = \left( \frac{\delta^2}{4k} \lim_{t \to 0} \frac{1}{t} \right) \left( 1 + \frac{2k}{\delta^2} \lim_{t \to 0} t \ln t \right)
\]

The last limit can be written as an indeterminate form \(\infty/\infty\).
\[
\lim_{t \to 0} t \ln t = \lim_{t \to 0} \ln t.
\]

Apply L'Hôpital's rule and differentiate the numerator and denominator.
\[
\lim_{t \to 0} t \ln t \xrightarrow{H} \lim_{t \to 0} \frac{\ln t}{\frac{1}{t}} = \lim_{t \to 0} (-t) = 0
\]

Thus,
\[
\lim_{t \to 0} \left( \frac{\delta^2}{4kt} + \ln \sqrt{t} \right) = \frac{\delta^2}{4k} \lim_{t \to 0} \frac{1}{t} \to \infty.
\]

This means that
\[
\lim_{t \to 0} S(\delta, t) \to \frac{1}{\sqrt{4\pi k}} e^{-\infty} = 0.
\]

Therefore, for any fixed \(\delta > 0\),
\[
\max_{\delta \leq |x| < \infty} S(x, t) \to 0 \quad \text{as } t \to 0.
\]