Exercise 4

Here is a direct relationship between the wave and diffusion equations. Let $u(x,t)$ solve the wave equation on the whole line with bounded second derivatives. Let

$$v(x,t) = \frac{c}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-s^2c^2/4kt} u(x,s)\,ds.$$

(a) Show that $v(x,t)$ solves the diffusion equation!

(b) Show that $\lim_{t\to 0} v(x,t) = u(x,0)$.

(*Hint:*

(a) Write the formula as $v(x,t) = \int_{-\infty}^{\infty} H(s,t)u(x,s)\,ds$, where $H(x,t)$ solves the diffusion equation with constant k/c^2 for $t > 0$. Then differentiate $v(x,t)$ using Section A.3.

(b) Use the fact that $H(s,t)$ is essentially the source function of the diffusion equation with the spatial variable s.)

Solution

Part (a)

Following the hint, make the substitution,

$$H(x,t) = \frac{c}{\sqrt{4\pi kt}} e^{-\frac{x^2c^2}{4kt}},$$

so that we can write $v(x,t)$ as

$$v(x,t) = \int_{-\infty}^{\infty} H(s,t)u(x,s)\,ds.$$

Let's check to see that $H(x,t)$ satisfies the diffusion equation.

$$\begin{aligned}
H_t &= \frac{c}{\sqrt{4\pi k}}\left[\left(-\frac{1}{2}t^{-\frac{3}{2}}\right)e^{-\frac{x^2c^2}{4kt}} + t^{-\frac{1}{2}}\left(\frac{x^2c^2}{4kt^2}\right)e^{-\frac{x^2c^2}{4kt}}\right] = -\frac{c}{4\sqrt{\pi kt^3}}e^{-\frac{x^2c^2}{4kt}}\left(1 - \frac{x^2c^2}{2kt}\right) \\
H_x &= \frac{c}{\sqrt{4\pi kt}}e^{-\frac{x^2c^2}{4kt}}\left(-\frac{xc^2}{2kt}\right) = -\frac{c^3}{4\sqrt{\pi k^3t^3}}xe^{-\frac{x^2c^2}{4kt}} \\
H_{xx} &= -\frac{c^3}{4\sqrt{\pi k^3t^3}}\left[e^{-\frac{x^2c^2}{4kt}} + x\left(-\frac{xc^2}{2kt}\right)e^{-\frac{x^2c^2}{4kt}}\right] = -\frac{c^3}{4\sqrt{\pi k^3t^3}}e^{-\frac{x^2c^2}{4kt}}\left(1 - \frac{x^2c^2}{2kt}\right)
\end{aligned}$$

Comparing H_t and H_{xx}, we see that $H_t = \frac{k}{c^2}H_{xx}$. Hence, H satisfies the diffusion equation with diffusion constant k/c^2. Now we will show that v satisfies the diffusion equation.

$$\begin{aligned}
v_t &= \int_{-\infty}^{\infty} H_t(s,t)u(x,s)\,ds = \frac{k}{c^2}\int_{-\infty}^{\infty} H_{xx}(s,t)u(x,s)\,ds \\
v_x &= \int_{-\infty}^{\infty} H(s,t)u_x(x,s)\,ds \\
v_{xx} &= \int_{-\infty}^{\infty} H(s,t)u_{xx}(x,s)\,ds
\end{aligned}$$

In order to move the x-derivatives onto u in the equation for v_t, proceed with the method of integration by parts. Note that H and H_x tend to 0 as $x \to \pm\infty$.

$$v_t = \frac{k}{c^2}\left[\underbrace{H_x(s,t)}_{\to 0}u(x,s)\bigg|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} H_x(s,t)u_x(x,s)\,ds\right] = -\frac{k}{c^2}\int_{-\infty}^{\infty} H_x(s,t)u_x(x,s)\,ds$$

Use integration by parts once more.

$$v_t = -\frac{k}{c^2}\left[\underbrace{H(s,t)}_{\to 0}u_x(x,s)\bigg|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} H(s,t)u_{xx}(x,s)\,ds\right] = \frac{k}{c^2}\int_{-\infty}^{\infty} H(s,t)u_{xx}(x,s)\,ds$$

Therefore, $v(x,t)$ satisfies the diffusion equation with diffusion constant k/c^2.

$$v_t = \frac{k}{c^2}v_{xx}$$

Part (b)

Following the hint, we will use the fact that $H(x,t)$ is the Green's function of the diffusion equation. What this means is that H not only satisfies the diffusion equation (with diffusion constant k/c^2), but also the special initial condition $H(x,0) = \delta(x)$. Recall that $\delta(x)$ is the Dirac delta function with the very handy "sifting" property. Thus,

$$\begin{aligned}
\lim_{t\to 0} v(x,t) &= \lim_{t\to 0}\int_{-\infty}^{\infty} H(s,t)u(x,s)\,ds \\
&= \int_{-\infty}^{\infty}\left[\lim_{t\to 0} H(s,t)\right]u(x,s)\,ds \\
&= \int_{-\infty}^{\infty} H(s,0)u(x,s)\,ds \\
&= \int_{-\infty}^{\infty} \delta(s)u(x,s)\,ds \\
&= u(x,0).
\end{aligned}$$

Proof that $H(s,0) = \delta(s)$

If we don't know to use the Dirac delta function for $H(s,0)$, we could deduce it by evaluating the limit with brute force. Let's do it.

$$\lim_{t\to 0} H(s,t) = \lim_{t\to 0}\frac{c}{\sqrt{4\pi kt}}e^{-\frac{s^2c^2}{4kt}} = \frac{c}{\sqrt{4\pi k}}\lim_{t\to 0}\frac{1}{\sqrt{t}}e^{-\frac{s^2c^2}{4kt}}$$

Plugging in $t = 0$ yields the indeterminate form $0/0$. However, applying L'Hôpital's rule doesn't lead to any simplification, so we have to try something else. Proceed by bringing $\sqrt{t}$ to the exponent of the exponential function.

$$\lim_{t\to 0} H(s,t) = \frac{c}{\sqrt{4\pi k}}\lim_{t\to 0} e^{-\frac{s^2c^2}{4kt} - \ln\sqrt{t}}$$

Factor out the minus sign and bring the limit into the exponent.

$$\lim_{t\to 0} H(s,t) = \frac{c}{\sqrt{4\pi k}} e^{-\lim\limits_{t\to 0}\left(\frac{s^2c^2}{4kt} + \ln\sqrt{t}\right)}$$

Plugging in $t = 0$ now yields the indeterminate form $\infty - \infty$, so factor out $s^2c^2/4kt$ in order to make this a product. Also, change $\sqrt{t}$ to t by bringing a factor of $1/2$ in front.

$$\lim_{t\to 0}\left(\frac{s^2c^2}{4kt} + \frac{1}{2}\ln t\right) = \lim_{t\to 0}\frac{s^2c^2}{4kt}\left(1 + \frac{2kt}{s^2c^2}\ln t\right)$$

Since the limit of a product is the product of the limits, we can write this as

$$\lim_{t\to 0}\left(\frac{s^2c^2}{4kt} + \frac{1}{2}\ln t\right) = \left(\lim_{t\to 0}\frac{s^2c^2}{4kt}\right)\left[\lim_{t\to 0}\left(1 + \frac{2kt}{s^2c^2}\ln t\right)\right].$$

The limit of a sum is the sum of the limits, so

$$\lim_{t\to 0}\left(\frac{s^2c^2}{4kt} + \frac{1}{2}\ln t\right) = \left(\lim_{t\to 0}\frac{s^2c^2}{4kt}\right)\left(\lim_{t\to 0} 1 + \lim_{t\to 0}\frac{2kt}{s^2c^2}\ln t\right).$$

Bring the constants in front of the limits.

$$\lim_{t\to 0}\left(\frac{s^2c^2}{4kt} + \frac{1}{2}\ln t\right) = \left(\frac{s^2c^2}{4k}\lim_{t\to 0}\frac{1}{t}\right)\left(1 + \frac{2k}{s^2c^2}\lim_{t\to 0} t\ln t\right).$$

The last limit can be written as an indeterminate form ∞/∞.

$$\lim_{t\to 0} t\ln t = \lim_{t\to 0}\frac{\ln t}{\frac{1}{t}}$$

Apply L'Hôpital's rule and differentiate the numerator and denominator.

$$\lim_{t\to 0} t\ln t \overset{\frac{\infty}{\infty}}{\underset{\text{H}}{=}} \lim_{t\to 0}\frac{\frac{1}{t}}{-\frac{1}{t^2}} = \lim_{t\to 0}(-t) = 0$$

Thus,

$$\lim_{t\to 0}\left(\frac{s^2c^2}{4kt} + \frac{1}{2}\ln t\right) = \left(\frac{s^2c^2}{4k}\lim_{t\to 0}\frac{1}{t}\right) \to \infty$$

unless s itself happens to go to 0 as well. We'll deal with this case in a moment, but for now we conclude that for s nonzero,

$$\lim_{t\to 0} H(s,t) \to \frac{c}{\sqrt{4\pi k}} e^{-\infty} = 0.$$

Consider now the case where $s \to 0$ along with $t \to 0$. Then

$$\lim_{\substack{t\to 0\\ s\to 0}}\left(\frac{s^2c^2}{4kt} + \frac{1}{2}\ln t\right) = \frac{c^2}{4k}\underbrace{\lim_{\substack{t\to 0\\ s\to 0}}\frac{s^2}{t}}_{\to\, 0} + \frac{1}{2}\underbrace{\lim_{\substack{t\to 0\\ s\to 0}}\ln t}_{\to\, -\infty} \to -\infty.$$

Hence,

$$\lim_{\substack{t\to 0\\ s\to 0}} H(s,t) \to \frac{c}{\sqrt{4\pi k}} e^{\infty} = \infty.$$

What we have shown is that

$$\lim_{t\to 0} H(s,t) = \delta(s) = \begin{cases} 0 & s \neq 0 \\ \infty & s = 0 \end{cases}.$$