Exercise 4

Here is a direct relationship between the wave and diffusion equations. Let $u(x,t)$ solve the wave equation on the whole line with bounded second derivatives. Let

$$v(x,t) = \frac{c}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-s^2c^2/4kt} u(x,s)\, ds.$$

(a) Show that $v(x,t)$ solves the diffusion equation!

(b) Show that $\lim_{t\to 0} v(x,t) = u(x,0)$.

(*Hint:*

(a) Write the formula as $v(x,t) = \int_{-\infty}^{\infty} H(s,t)u(x,s)\, ds$, where $H(x,t)$ solves the diffusion equation with constant k/c^2 for $t > 0$. Then differentiate $v(x,t)$ using Section A.3.

(b) Use the fact that $H(s,t)$ is essentially the source function of the diffusion equation with the spatial variable s.)