Exercise 1

Show that there is no maximum principle for the wave equation.

Solution

We can prove this using the method of proof by contradiction. Suppose that there is a maximum principle for the wave equation $u_{tt} = c^2 u_{xx}$. Let $u(x,t)$ be a solution to the wave equation in a rectangle $0 \leq x \leq L$ and $0 \leq t \leq T$. Then the maximum of $u(x,t)$ is assumed either initially $(t = 0)$ or on the lateral sides $(x = 0$ or $x = L)$. Consider a particular solution,

$$u(x,t) = t + 1 - (x - ct)^2.$$

We can show that this satisfies the wave equation.

$$\begin{aligned} u_x &= -2(x - ct) \\ u_{xx} &= -2 \\ u_t &= 1 - 2(x - ct) \cdot (-c) = 1 + 2cx - 2c^2 t \\ u_{tt} &= -2c^2 \end{aligned}$$

Hence, $u_{tt} = c^2 u_{xx}$. According to the maximum principle, the maximum of $u(x,t)$ is $u(0,0) = 1$. The maximum can also be determined by calculating the first derivative with respect to x and setting it equal to 0.

$$u_x = -2(x - ct) = 0$$

Solving for x yields $x = ct$. This implies that the maximum is moving to the right with speed c. To determine what the maximum is, substitute ct for x in $u(x,t)$.

$$u(x = ct, t) = t + 1$$

What this indicates is that the maximum grows linearly in time. It is here where the contradiction lies. The maximum principle tells us that the maximum of $u(x,t)$ is 1, but we just concluded that the maximum of u is bigger than 1 for any $t > 0$. Therefore, there is no maximum principle for the wave equation.