Exercise 2

Consider a traveling wave \( u(x, t) = f(x - at) \) where \( f \) is a given function of one variable.

(a) If it is a solution of the wave equation, show that the speed must be \( a \pm c \) (unless \( f \) is a linear function).

(b) If it is a solution of the diffusion equation, find \( f \) and show that the speed \( a \) is arbitrary.

Solution

Part (a)

If \( u(x, t) = f(x - at) \) is a solution to the wave equation, then it has to satisfy \( u_{tt} = c^2 u_{xx} \).

\[
\begin{align*}
    u_t &= (-a) \cdot f' \\
    u_{tt} &= a^2 \cdot f'' \\
    u_x &= f' \\
    u_{xx} &= f''
\end{align*}
\]

Substituting these expressions for the terms in the PDE yields

\[
a^2 f'' = c^2 f''.
\]

This implies that

\[
a^2 = c^2
\]

or

\[
a = \pm c.
\]

If \( f(x - at) \) is linear, then \( f'' = 0 \) and \( a^2 \) need not equal \( c^2 \).

Part (b)

If \( u(x, t) = f(x - at) \) is a solution to the diffusion equation, then it has to satisfy \( u_t = ku_{xx} \).

\[
\begin{align*}
    u_t &= (-a) \cdot f' \\
    u_x &= f' \\
    u_{xx} &= f''
\end{align*}
\]

Substituting these expressions for the terms in the PDE yields

\[-af' = kf''.
\]

Rewrite this as

\[-\frac{a}{k} = \frac{f''}{f'} = \frac{d\ln f'}{d\xi},
\]

where \( \xi = x - at \). Integrate both sides once with respect to \( \xi \).

\[-\frac{a}{k} \xi + C = \ln f'
\]

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Exponentiate both sides.
\[ e^{-\frac{a}{k} \xi} + C = f' \]

Introduce a new constant of integration \( C_1 = e^C \).
\[ C_1 e^{-\frac{a}{k} \xi} = f' \]

Integrate both sides with respect to \( \xi \) a second time.
\[ f(\xi) = C_3 e^{-\frac{a}{k} \xi} + C_2, \]

where \( C_2 \) and \( C_3 = -kC_1/a \) are other arbitrary constants. Now change back to the original variables, \( x \) and \( t \).
\[ f(x - at) = C_3 e^{-\frac{a}{k}(x - at)} + C_2 \tag{1} \]

We can check that this satisfies the diffusion equation.
\[
\begin{align*}
  u_t &= C_3 e^{-\frac{a}{k}(x - at)} \cdot \left( \frac{a^2}{k} \right) \\
  u_x &= C_3 e^{-\frac{a}{k}(x - at)} \cdot \left( -\frac{a}{k} \right) \\
  u_{xx} &= C_3 e^{-\frac{a}{k}(x - at)} \cdot \left( \frac{a^2}{k^2} \right)
\end{align*}
\]

Therefore, \( u_t = ku_{xx} \), which means (1) is the correct solution for \( f \). Note that there are no restrictions on \( a \); that is, it is arbitrary.