Exercise 3

Let u satisfy the diffusion equation $u_t = \frac{1}{2}u_{xx}$. Let

$$v(x,t) = \frac{1}{\sqrt{t}}e^{x^2/2t}u\left(\frac{x}{t},\frac{1}{t}\right).$$

Show that v satisfies the "backward" diffusion equation $v_t = -\frac{1}{2}v_{xx}$ for $t > 0$. [TYPO: This should be u, not v.]

Solution

All that we have to do here is take derivatives of the given function $v(x,t)$ and then show that $v_t = -\frac{1}{2}v_{xx}$ for $t > 0$. Somewhere along the way we'll make use of the fact that $u_t = \frac{1}{2}u_{xx}$.

$$\begin{aligned}
v_t &= \left[-\frac{1}{2}t^{-\frac{3}{2}}e^{\frac{x^2}{2t}} + t^{-\frac{1}{2}}\left(-\frac{x^2}{2t^2}\right)e^{\frac{x^2}{2t}}\right]u + t^{-\frac{1}{2}}e^{\frac{x^2}{2t}}\left[\left(-\frac{x}{t^2}\right)u_x + \left(-\frac{1}{t^2}\right)u_t\right] \\
v_x &= t^{-\frac{1}{2}}\left(\frac{x}{t}\right)e^{\frac{x^2}{2t}}u + t^{-\frac{1}{2}}e^{\frac{x^2}{2t}}\left(\frac{1}{t}\right)u_x = t^{-\frac{3}{2}}e^{\frac{x^2}{2t}}(xu + u_x) \\
v_{xx} &= t^{-\frac{3}{2}}\left\{\left(\frac{x}{t}\right)e^{\frac{x^2}{2t}}(xu + u_x) + e^{\frac{x^2}{2t}}\left[u + x\left(\frac{1}{t}\right)u_x + \left(\frac{1}{t}\right)u_{xx}\right]\right\}
\end{aligned}$$

The expressions for v_t and v_{xx} can be simplified as follows.

$$\begin{aligned}
v_t &= -\frac{1}{2}t^{-\frac{5}{2}}e^{\frac{x^2}{2t}}\left[(x^2+t)u + 2xu_x + 2u_t\right] \\
v_{xx} &= t^{-\frac{5}{2}}e^{\frac{x^2}{2t}}\left[(x^2+t)u + 2xu_x + u_{xx}\right]
\end{aligned}$$

Now plug in $u_t = \frac{1}{2}u_{xx}$ in the equation for v_t.

$$v_t = -\frac{1}{2}t^{-\frac{5}{2}}e^{\frac{x^2}{2t}}\left[(x^2+t)u + 2xu_x + u_{xx}\right]$$

Therefore, v satisfies the "backward" diffusion equation for $t > 0$.

$$v_t = -\frac{1}{2}v_{xx}$$