Exercise 1

Solve $u_t = ku_{xx}$; $u(x,0) = e^{-x}$; $u(0,t) = 0$ on the half-line $0 < x < \infty$.

Solution

Solution by the Method of Reflection

To solve the diffusion equation on the half-line with $u(0,t) = 0$, consider the same problem over the whole line, using the odd extension of the initial condition $u(x,0)$.

$$v_t = kv_{xx}, \quad -\infty < x < \infty,\ t > 0$$

$$v(x,0) = \phi_{\text{odd}}(x) = \begin{cases} e^{-x} & x > 0 \\ -e^{x} & x < 0 \end{cases}$$

The solution to v is given in section 2.4 on page 49.

$$v(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} \exp\left[-\frac{(x-r)^2}{4kt}\right] \phi_{\text{odd}}(r)\, dr$$

The solution to u is then just the restriction of v to $x > 0$.

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} \exp\left[-\frac{(x-r)^2}{4kt}\right] \phi_{\text{odd}}(r)\, dr, \quad x > 0$$

Our task now is to simplify this formula. Split up the integral into two—one over the negative values of x and one over the positive values of x—and substitute the appropriate functions of $\phi_{\text{odd}}(x)$ in these intervals.

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}} \left\{ \int_{-\infty}^{0} \exp\left[-\frac{(x-r)^2}{4kt}\right] (-e^{r})\, dr + \int_{0}^{\infty} \exp\left[-\frac{(x-r)^2}{4kt}\right] (e^{-r})\, dr \right\}$$

Substitute $r = -p$ in the first integral and $r = p$ in the second integral to get

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}} \left\{ \int_{\infty}^{0} \exp\left[-\frac{(x+p)^2}{4kt}\right] e^{-p}\, dp + \int_{0}^{\infty} \exp\left[-\frac{(x-p)^2}{4kt}\right] e^{-p}\, dp \right\}.$$

Combine the exponential functions.

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}} \left\{ \int_{\infty}^{0} \exp\left[-\frac{(x+p)^2}{4kt} - p\right] dp + \int_{0}^{\infty} \exp\left[-\frac{(x-p)^2}{4kt} - p\right] dp \right\}$$

We can rewrite the exponents as follows.

$$\begin{aligned}
-\frac{(x+p)^2}{4kt} - p &= \frac{-x^2 - 2xp - p^2 - 4ktp}{4kt} \\
&= \frac{-x^2 - (2x+4kt)p - p^2}{4kt} \\
&= \frac{-x^2 + (x+2kt)^2 - (x+2kt)^2 - (2x+4kt)p - p^2}{4kt} \\
&= \frac{-x^2 + (x+2kt)^2}{4kt} - \frac{(x+2kt)^2 + 2(x+2kt)p + p^2}{4kt} \\
&= kt + x - \frac{(x+2kt+p)^2}{4kt}
\end{aligned}$$

$$\begin{aligned}
-\frac{(x-p)^2}{4kt} - p &= \frac{-x^2 + 2xp - p^2 - 4ktp}{4kt} \\
&= \frac{-x^2 + (2x-4kt)p - p^2}{4kt} \\
&= \frac{-x^2 + (x-2kt)^2 - (x-2kt)^2 + (2x-4kt)p - p^2}{4kt} \\
&= \frac{-x^2 + (x-2kt)^2}{4kt} - \frac{(x-2kt)^2 - 2(x-2kt)p + p^2}{4kt} \\
&= kt - x - \frac{(x-2kt-p)^2}{4kt}
\end{aligned}$$

Substitute these new expressions into the integrals.

$$\begin{aligned}
u(x,t) &= \frac{1}{\sqrt{4\pi kt}}\left\{\int_{\infty}^{0} \exp\left[kt + x - \frac{(x+2kt+p)^2}{4kt}\right] dp + \int_0^{\infty} \exp\left[kt - x - \frac{(x-2kt-p)^2}{4kt}\right] dp\right\} \\
&= \frac{1}{\sqrt{4\pi kt}}\left\{e^{kt+x}\int_{\infty}^{0} \exp\left[-\frac{(x+2kt+p)^2}{4kt}\right] dp + e^{kt-x}\int_0^{\infty} \exp\left[-\frac{(x-2kt-p)^2}{4kt}\right] dp\right\}
\end{aligned}$$

Now make the following substitutions.

$$\begin{aligned}
q &= \frac{x+2kt+p}{\sqrt{4kt}} & \qquad w &= \frac{-x+2kt+p}{\sqrt{4kt}} \\
dq &= \frac{dp}{\sqrt{4kt}} & \qquad dw &= \frac{dp}{\sqrt{4kt}}
\end{aligned}$$

The solution becomes

$$\begin{aligned}
u(x,t) &= \frac{1}{\sqrt{\pi}}\left(e^{kt+x}\int_{\infty}^{\frac{x+2kt}{\sqrt{4kt}}} e^{-q^2}\,dq + e^{kt-x}\int_{\frac{-x+2kt}{\sqrt{4kt}}}^{\infty} e^{-w^2}\,dw\right) \\
&= \frac{e^{kt+x}}{2}\frac{2}{\sqrt{\pi}}\int_{\infty}^{\frac{x+2kt}{\sqrt{4kt}}} e^{-q^2}\,dq + \frac{e^{kt-x}}{2}\frac{2}{\sqrt{\pi}}\int_{\frac{-x+2kt}{\sqrt{4kt}}}^{\infty} e^{-w^2}\,dw \\
&= -\frac{e^{kt+x}}{2}\left(\frac{2}{\sqrt{\pi}}\int_{\frac{x+2kt}{\sqrt{4kt}}}^{\infty} e^{-q^2}\,dq\right) + \frac{e^{kt-x}}{2}\left(\frac{2}{\sqrt{\pi}}\int_{\frac{-x+2kt}{\sqrt{4kt}}}^{\infty} e^{-w^2}\,dw\right).
\end{aligned}$$

In each set of parentheses is how a special function known as the complementary error function is defined.

$$u(x,t) = -\frac{e^{kt+x}}{2}\operatorname{erfc}\left(\frac{x+2kt}{\sqrt{4kt}}\right) + \frac{e^{kt-x}}{2}\operatorname{erfc}\left(\frac{-x+2kt}{\sqrt{4kt}}\right)$$

Therefore,

$$u(x,t) = \frac{1}{2}\left[e^{kt-x}\operatorname{erfc}\left(\frac{-x+2kt}{\sqrt{4kt}}\right) - e^{kt+x}\operatorname{erfc}\left(\frac{x+2kt}{\sqrt{4kt}}\right)\right].$$

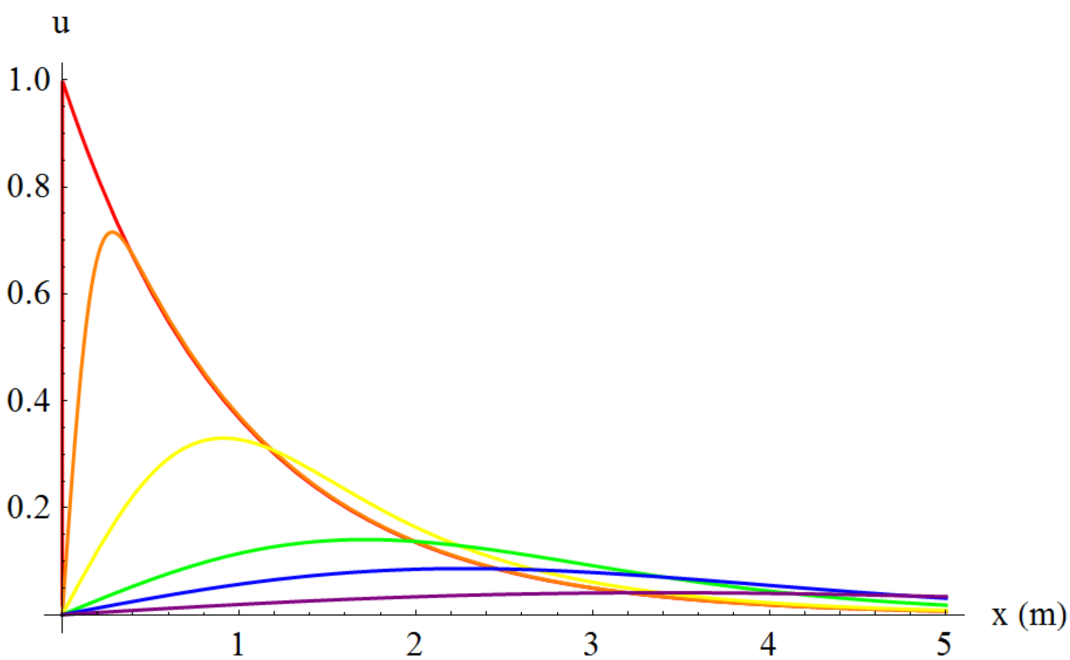

Figure 1: This is a plot of $u(x,t)$ versus x with $k = 1$ m^2/s for six different times: $t = 0$ s (in red), $t = 0.01$ s (in orange), $t = 0.2$ s (in yellow), $t = 1$ s (in green), $t = 2$ s (in blue), and $t = 5$ s (in purple).