# Exercise 2

Solve $u_t = ku_{xx}$; $u(x,0) = 0$; $u(0,t) = 1$ on the half-line $0 < x < \infty$.

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### Solution

In order to apply the method of reflection, the Dirichlet boundary condition needs to be homogeneous. Make it so with the substitution,

$$U(x,t) = u(x,t) - 1.$$

Find the derivatives of $u$ in terms of this new variable.

$$\begin{aligned} U_t &= u_t \\ U_x &= u_x \\ U_{xx} &= u_{xx} \end{aligned}$$

As a result, $U$ satisfies the diffusion equation as well.

$$U_t = kU_{xx}, \quad 0 < x < \infty,\ t > 0$$

The initial and boundary conditions associated with it are

$$\begin{aligned} U(x,0) &= u(x,0) - 1 = 0 - 1 = -1 \\ U(0,t) &= u(0,t) - 1 = 1 - 1 = 0. \end{aligned}$$

Use the method of reflection to solve for $U$. Consider the same problem over the whole line, using the odd extension of $U(x,0)$ to satisfy the Dirichlet boundary condition. Let $V = V(x,t)$ be the solution to this problem.

$$V_t = kV_{xx}, \quad -\infty < x < \infty,\ t > 0$$

$$V(x,0) = \phi_{\text{odd}}(x) = \begin{cases} -1 & x > 0 \\ 1 & x < 0 \end{cases}$$

The solution to $V$ is given in section 2.4 on page 49.

$$V(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} \exp\left[-\frac{(x-r)^2}{4kt}\right] \phi_{\text{odd}}(r)\, dr$$

The solution to $U$ is then just the restriction of $V$ to $x > 0$.

$$U(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} \exp\left[-\frac{(x-r)^2}{4kt}\right] \phi_{\text{odd}}(r)\, dr, \quad x > 0$$

Our task now is to simplify this formula. Split up the integral into two—one over the negative values of $x$ and one over the positive values of $x$—and substitute the appropriate functions of $\phi_{\text{odd}}(x)$ in these intervals.

$$U(x,t) = \frac{1}{\sqrt{4\pi kt}} \left\{ \int_{-\infty}^{0} \exp\left[-\frac{(x-r)^2}{4kt}\right] (1)\, dr + \int_{0}^{\infty} \exp\left[-\frac{(x-r)^2}{4kt}\right] (-1)\, dr \right\}, \quad x > 0$$

Let $r = -p$ in the first integral and let $r = p$ in the second integral.

$$\begin{aligned} U(x,t) &= \frac{1}{\sqrt{4\pi kt}} \left\{ \int_{\infty}^{0} \exp\left[-\frac{(x+p)^2}{4kt}\right] (1)(-dp) + \int_{0}^{\infty} \exp\left[-\frac{(x-p)^2}{4kt}\right] (-1)\, dp \right\}, \quad x > 0 \\ &= \frac{1}{\sqrt{4\pi kt}} \left\{ \int_{0}^{\infty} \exp\left[-\frac{(x+p)^2}{4kt}\right] dp - \int_{0}^{\infty} \exp\left[-\frac{(x-p)^2}{4kt}\right] dp \right\}, \quad x > 0 \end{aligned}$$

Now make the following substitutions.

$$\begin{aligned} q &= \frac{x+p}{\sqrt{4kt}} & w &= \frac{-x+p}{\sqrt{4kt}} \\ dq &= \frac{dp}{\sqrt{4kt}} & dw &= \frac{dp}{\sqrt{4kt}} \end{aligned}$$

The solution becomes

$$\begin{aligned} U(x,t) &= \frac{1}{\sqrt{\pi}} \left( \int_{\frac{x}{\sqrt{4kt}}}^{\infty} e^{-q^2}\, dq - \int_{-\frac{x}{\sqrt{4kt}}}^{\infty} e^{-w^2}\, dw \right) \\ &= -\frac{1}{\sqrt{\pi}} \int_{-\frac{x}{\sqrt{4kt}}}^{\frac{x}{\sqrt{4kt}}} e^{-q^2}\, dq \\ &= -\frac{2}{\sqrt{\pi}} \int_{0}^{\frac{x}{\sqrt{4kt}}} e^{-q^2}\, dq \\ &= -\operatorname{erf}\left(\frac{x}{\sqrt{4kt}}\right), \end{aligned}$$

where erf is a special function known as the error function. Therefore, since $u(x,t) = U(x,t) + 1$,

$$u(x,t) = 1 - \operatorname{erf}\left(\frac{x}{\sqrt{4kt}}\right) = \operatorname{erfc}\left(\frac{x}{\sqrt{4kt}}\right),$$

where erfc is a special function known as the complementary error function.

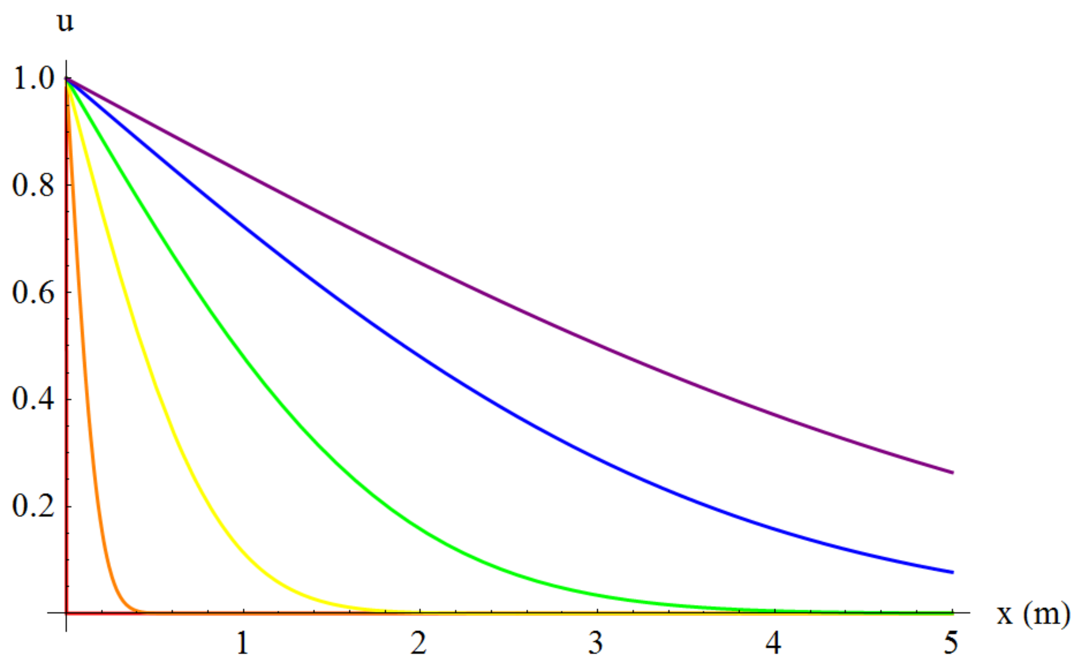


Figure 1: This is a plot of $u(x,t)$ versus $x$ with $k = 1$ m$^2$/s for six different times: $t = 0$ s (in red), $t = 0.01$ s (in orange), $t = 0.2$ s (in yellow), $t = 1$ s (in green), $t = 4$ s (in blue), and $t = 10$ s (in purple). Note that the dimensionless temperature (or concentration) tends to 1 as $t \to \infty$.