Exercise 4

Consider the following problem with a Robin boundary condition:

DE:	$u_t = ku_{xx}$	on the half-line $0 < x < \infty$ (and $0 < t < \infty$)	
IC:	$u(x,0) = x$	for $t = 0$ and $0 < x < \infty$	$(*)$
BC:	$u_x(0,t) - 2u(0,t) = 0$	for $x = 0$	

The purpose of this exercise is to verify the solution formula for (*). Let $f(x) = x$ for $x > 0$, let $f(x) = x + 1 - e^{2x}$ for $x < 0$, and let

$$v(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-(x-y)^2/4kt} f(y)\, dy.$$

(a) What PDE and initial condition does $v(x,t)$ satisfy for $-\infty < x < \infty$?

(b) Let $w = v_x - 2v$. What PDE and initial condition does $w(x,t)$ satisfy for $-\infty < x < \infty$?

(c) Show that $f'(x) - 2f(x)$ is an odd function (for $x \neq 0$).

(d) Use Exercise 2.4.11 to show that w is an odd function of x.

(e) Deduce that $v(x,t)$ satisfies (*) for $x > 0$. Assuming uniqueness, deduce that the solution of (*) is given by

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-(x-y)^2/4kt} f(y)\, dy.$$