Exercise 3

Derive the solution formula for the half-line Neumann problem $w_t - kw_{xx} = 0$ for $0 < x < \infty$, $0 < t < \infty$; $w_x(0, t) = 0$; $w(x, 0) = \phi(x)$.

Solution

Solution by the Method of Reflection

To solve the diffusion equation on the half-line with $w_x(0, t) = 0$, consider the same problem over the whole line, using the even extension of the initial condition $\phi(x)$.

$$W_t = kW_{xx}, \quad -\infty < x < \infty,\ t > 0$$
$$W(x, 0) = \phi_{\text{even}}(x) = \begin{cases} \phi(x) & x > 0 \\ \phi(-x) & x < 0 \end{cases}$$

The solution to W is given in section 2.4 on page 49.

$$W(x, t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} \exp\left[-\frac{(x-r)^2}{4kt}\right] \phi_{\text{even}}(r)\, dr$$

The solution to w is then just the restriction of W to $x > 0$.

$$w(x, t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} \exp\left[-\frac{(x-r)^2}{4kt}\right] \phi_{\text{even}}(r)\, dr, \quad x > 0$$

Our task now is to simplify this formula. Split up the integral into two—one over the negative values of x and one over the positive values of x—and substitute the appropriate functions of $\phi_{\text{even}}(x)$ in these intervals.

$$w(x, t) = \frac{1}{\sqrt{4\pi kt}} \left\{ \int_{-\infty}^{0} \exp\left[-\frac{(x-r)^2}{4kt}\right] [\phi(-r)]\, dr + \int_{0}^{\infty} e^{-\frac{(x-r)^2}{4kt}} [\phi(r)]\, dr \right\}, \quad x > 0$$

Substitute $r = -p$ in the first integral and $r = p$ in the second integral.

$$w(x, t) = \frac{1}{\sqrt{4\pi kt}} \left\{ \int_{\infty}^{0} \exp\left[-\frac{(x+p)^2}{4kt}\right] \phi(p)(-dp) + \int_{0}^{\infty} \exp\left[-\frac{(x-p)^2}{4kt}\right] \phi(p)\, dp \right\}, \quad x > 0$$

Use the minus sign to switch the limits of the first integral.

$$w(x, t) = \frac{1}{\sqrt{4\pi kt}} \left\{ \int_{0}^{\infty} \exp\left[-\frac{(x+p)^2}{4kt}\right] \phi(p)\, dp + \int_{0}^{\infty} \exp\left[-\frac{(x-p)^2}{4kt}\right] \phi(p)\, dp \right\}, \quad x > 0$$

Therefore,

$$w(x, t) = \frac{1}{\sqrt{4\pi kt}} \int_{0}^{\infty} \left\{ \exp\left[-\frac{(x+p)^2}{4kt}\right] + \exp\left[-\frac{(x-p)^2}{4kt}\right] \right\} \phi(p)\, dp, \quad x > 0.$$

This can be written compactly as

$$w(x, t) = \int_{0}^{\infty} [G(x+p, t) + G(x-p, t)]\phi(p)\, dp, \quad x > 0,$$

where $G = G(x,t)$ is the Green's function for the one-dimensional diffusion equation.

$$G(x,t) = \frac{1}{\sqrt{4\pi kt}} \exp\left(-\frac{x^2}{4kt}\right)$$

Solution by the Substitution Method

If we didn't know about the properties of even and odd functions, we could still solve the Neumann problem on the half-line by changing it to the Dirichlet problem with a substitution. The boundary condition is $w_x(0,t) = 0$, so to change it to a Dirichlet boundary condition, we'll make the substitution $v(x,t) = w_x(x,t)$. Differentiate both sides of the diffusion equation and the initial condition with respect to x.

$$w_{tx} - kw_{xxx} = 0, \quad w_x(x,0) = \phi'(x)$$

Group the derivatives as follows.

$$(w_x)_t - k(w_x)_{xx} = 0, \quad w_x(x,0) = \phi'(x)$$

Now apply the substitution $v = w_x$.

$$v_t - kv_{xx} = 0, \quad v(x,0) = \phi'(x)$$

The boundary condition becomes $v(0,t) = 0$, which is exactly what we need in order to use the formula for the Dirichlet problem.

$$v(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_0^\infty \left[e^{-\frac{(x-s)^2}{4kt}} - e^{-\frac{(x+s)^2}{4kt}}\right] \phi'(s)\,ds$$

We don't want there to be a prime symbol on ϕ, so we move it to the term in square brackets by using integration by parts.

$$v(x,t) = \frac{1}{\sqrt{4\pi kt}} \left\{ \left[e^{-\frac{(x-s)^2}{4kt}} - e^{-\frac{(x+s)^2}{4kt}}\right] \phi(s)\bigg|_0^\infty - \int_0^\infty \frac{\partial}{\partial s}\left[e^{-\frac{(x-s)^2}{4kt}} - e^{-\frac{(x+s)^2}{4kt}}\right] \phi(s)\,ds \right\}$$

The first term in the curly braces evaluates to 0. In the second term, evaluate the derivative of the expression in square brackets.

$$v(x,t) = -\frac{1}{\sqrt{4\pi kt}} \int_0^\infty \left[\frac{x-s}{2kt} e^{-\frac{(x-s)^2}{4kt}} + \frac{x+s}{2kt} e^{-\frac{(x+s)^2}{4kt}}\right] \phi(s)\,ds$$

Now that we solved for v, we plug back in w_x and integrate the result with respect to x to determine w.

$$w(x,t) = \int^x \left\{ -\frac{1}{\sqrt{4\pi kt}} \int_0^\infty \left[\frac{r-s}{2kt} e^{-\frac{(r-s)^2}{4kt}} + \frac{r+s}{2kt} e^{-\frac{(r+s)^2}{4kt}}\right] \phi(s)\,ds \right\} dr + C,$$

where C is a constant of integration we'll figure out later. Bring the integral inside.

$$w(x,t) = -\frac{1}{\sqrt{4\pi kt}} \int_0^\infty \left[\int^x \frac{r-s}{2kt} e^{-\frac{(r-s)^2}{4kt}}\,dr + \int^x \frac{r+s}{2kt} e^{-\frac{(r+s)^2}{4kt}}\,dr\right] \phi(s)\,ds + C$$

To solve the integrals, make the substitutions below.

$$p = -\frac{(r-s)^2}{4kt} \qquad q = -\frac{(r+s)^2}{4kt}$$
$$dp = -\frac{r-s}{2kt}\,dr \qquad dq = -\frac{r+s}{2kt}\,dr$$

$$w(x,t) = -\frac{1}{\sqrt{4\pi kt}}\int_0^\infty \left[-\int^{-\frac{(x-s)^2}{4kt}} e^p\,dp - \int^{-\frac{(x+s)^2}{4kt}} e^q\,dq\right]\phi(s)\,ds + C.$$

Evaluating the two integrals gives us

$$w(x,t) = \frac{1}{\sqrt{4\pi kt}}\int_0^\infty \left[e^{-\frac{(x-s)^2}{4kt}} + e^{-\frac{(x+s)^2}{4kt}}\right]\phi(s)\,ds + C.$$

Since we just want $w(x,0) = \phi(x)$, we set $C = 0$. Therefore,

$$w(x,t) = \frac{1}{\sqrt{4\pi kt}}\int_0^\infty \left[e^{-\frac{(x-s)^2}{4kt}} + e^{-\frac{(x+s)^2}{4kt}}\right]\phi(s)\,ds,$$

which matches what we got before with the method of reflection.