Exercise 10

Solve $u_{tt} = 9u_{xx}$ in $0 < x < \pi/2$, $u(x,0) = \cos x$, $u_t(x,0) = 0$, $u_x(0,t) = 0$, $u(\pi/2,t) = 0$.

Solution

The method of reflection will be used to solve this problem. Consider the corresponding problem over the whole line, using the extensions for the initial data that are even with respect to $x = 0$ and odd with respect to $x = \pi/2$ in order to satisfy the homogeneous boundary conditions.

$$\begin{aligned} &v_{tt} = 9v_{xx} \qquad -\infty < x < \infty,\ t > 0 \\ &v(x,0) = \phi_{\text{ext}}(x) \\ &v_t(x,0) = \psi_{\text{ext}}(x) \end{aligned}$$

$\phi_{\text{ext}}(x) = \cos x$ and $\psi_{\text{ext}}(x) = 0$ are suitable functions for the job. The solution for v is given by d'Alembert's formula.

$$\begin{aligned} v(x,t) &= \frac{1}{2}[\phi_{\text{ext}}(x+3t) + \phi_{\text{ext}}(x-3t)] + \frac{1}{2(3)}\int_{x-3t}^{x+3t} \psi_{\text{ext}}(s)\,ds \\ &= \frac{1}{2}[\cos(x+3t) + \cos(x-3t)] \\ &= \frac{1}{2}[\cos x\cos 3t - \cancel{\sin x \sin 3t} + \cos x \cos 3t + \cancel{\sin x \sin 3t}] \\ &= \cos x \cos 3t \end{aligned}$$

u is obtained by restricting this solution to $0 < x < \pi/2$. Therefore,

$$u(x,t) = \cos x \cos 3t, \quad 0 < x < \frac{\pi}{2}.$$