Exercise 6

Solve $u_{tt} = c^2 u_{xx}$ in $0 < x < \infty$, $0 \leq t < \infty$, $u(x, 0) = 0$, $u_t(x, 0) = V$,

$$u_t(0, t) + a u_x(0, t) = 0,$$

where V, a, and c are positive constants and $a > c$.

Solution

In order to apply the method of reflection to solve the wave equation on $0 < x < \infty$, the boundary condition has to be either Dirichlet or Neumann. It can be made Dirichlet by making the following change of variables.

$$v(x, t) = u_t(x, t) + a u_x(x, t)$$

Consequently, the boundary condition for v is

$$v(0, t) = u_t(0, t) + a u_x(0, t) = 0$$

as desired. Any derivative of a solution to the wave equation is also a solution, and any linear combination of solutions to the wave equation is also a solution. We suspect then that v satisfies the wave equation.

$$v_{tt} \stackrel{?}{=} c^2 v_{xx}$$

$$u_{ttt} + a u_{ttx} \stackrel{?}{=} c^2 (u_{xxt} + a u_{xxx})$$

$$u_{ttt} - c^2 u_{xxt} + a u_{ttx} - a c^2 u_{xxx} \stackrel{?}{=} 0$$

Provided that u_{xxt} and u_{ttx} are continuous in the xt-quarter-plane, the order of differentiation is immaterial by Clairaut's theorem.

$$\frac{\partial}{\partial t}(\underbrace{u_{tt} - c^2 u_{xx}}_{=0}) + a \frac{\partial}{\partial x}(\underbrace{u_{tt} - c^2 u_{xx}}_{=0}) \stackrel{?}{=} 0$$

$$0 = 0$$

Thus, v satisfies the wave equation.

$$v_{tt} = c^2 v_{xx}, \quad 0 < x < \infty, \ t \geq 0$$

The initial conditions for v are obtained from those for u, $u(x, 0) = 0$ and $u_t(x, 0) = V$.

$$v(x, 0) = u_t(x, 0) + a u_x(x, 0) = V + \underbrace{a \frac{\partial}{\partial x}[u(x, 0)]}_{=0} = V$$

$$v_t(x, 0) = u_{tt}(x, 0) + a u_{tx}(x, 0) = c^2 u_{xx}(x, 0) + \underbrace{a \frac{\partial}{\partial x}[u_t(x, 0)]}_{=0} = c^2 \frac{\partial^2}{\partial x^2}[u(x, 0)] = 0$$

u_{tx} and u_{xt} have been assumed to be continuous as well so that $u_{tx} = u_{xt}$. The method of reflection can be applied to solve for v. Consider the same problem over the whole line, where the odd extension of $v(x, 0)$ is used in order to satisfy the Dirichlet boundary condition at $x = 0$.

$$w_{tt} = c^2 w_{xx}, \quad -\infty < x < \infty,\ t \geq 0$$
$$w(x,0) = \phi_{\text{odd}}(x) = \begin{cases} V & \text{if } x > 0 \\ -V & \text{if } x < 0 \end{cases}, \quad w_t(x,0) = 0$$

The solution for w is given by d'Alembert's formula in section 2.1 on page 36.

$$w(x,t) = \frac{1}{2}[\phi_{\text{odd}}(x+ct) + \phi_{\text{odd}}(x-ct)]$$

The solution for v is then just the restriction of w to $x > 0$.

$$v(x,t) = \frac{1}{2}[\phi_{\text{odd}}(x+ct) + \phi_{\text{odd}}(x-ct)], \quad x > 0$$

Our task now is to write this formula in terms of the given constant V. Note that

$$\phi_{\text{odd}}(x+ct) = \begin{cases} V & \text{if } x+ct > 0 \\ -V & \text{if } x+ct < 0 \end{cases} \quad \text{and} \quad \phi_{\text{odd}}(x-ct) = \begin{cases} V & \text{if } x-ct > 0 \\ -V & \text{if } x-ct < 0 \end{cases},$$

so for every region in the xt-quarter-plane, we have to test whether $x - ct$ and $x + ct$ are greater than or less than zero. The characteristic curve $x - ct = 0$ is the line that separates the regions. They are illustrated below in Figure 1.

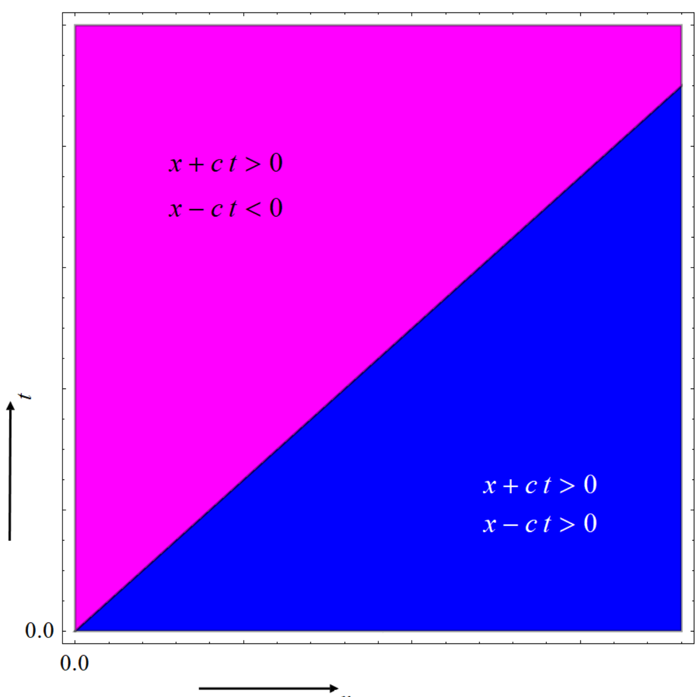

Figure 1: This figure illustrates the regions in the xt-quarter-plane that come about from using the odd extension of $v(x, 0)$. The solution for v has to be considered in each one.

The Magenta Region

In the magenta region $x + ct > 0$ and $x - ct < 0$, so the solution for v is

$$\begin{aligned} v(x,t) &= \frac{1}{2}[\phi_{\text{odd}}(x+ct) + \phi_{\text{odd}}(x-ct)] \\ &= \frac{1}{2}[V + (-V)] \\ &= 0. \end{aligned}$$

The Blue Region

In the blue region $x + ct > 0$ and $x - ct > 0$, so the solution for v is

$$\begin{aligned} v(x,t) &= \frac{1}{2}[\phi_{\text{odd}}(x+ct) + \phi_{\text{odd}}(x-ct)] \\ &= \frac{1}{2}(V + V) \\ &= V. \end{aligned}$$

As a result,

$$v(x,t) = \begin{cases} 0 & \text{if } x - ct < 0 \\ V & \text{if } x - ct > 0 \end{cases}.$$

Now that v is known, we can solve for u with the substitution made in the beginning.

$$u_t + au_x = \begin{cases} 0 & \text{if } x - ct < 0 \\ V & \text{if } x - ct > 0 \end{cases}$$

There are two first-order PDEs to solve, one for the magenta region and one for the blue region. The method of characteristics can be applied to solve them. On the paths in the xt-quarter-plane defined by

$$\frac{dx}{dt} = a, \tag{1}$$

the PDEs reduce to ODEs.

$$\frac{du}{dt} = \begin{cases} 0 & \text{if } x - ct < 0 \\ V & \text{if } x - ct > 0 \end{cases}$$

Integrate both sides of equation (1) with respect to t to obtain the characteristic curves,

$$x = at + \xi,$$

where ξ is the characteristic coordinate. Solving the equation for it gives

$$\xi = x - at.$$

Now integrate the two ODEs for u with respect to t.

$$u(\xi,t) = \begin{cases} f(\xi) & \text{if } x - ct < 0 \\ Vt + g(\xi) & \text{if } x - ct > 0 \end{cases},$$

where f and g are arbitrary functions to be determined. Consequently,

$$u(x,t) = \begin{cases} f(x-at) & \text{if } x-ct<0 \\ Vt + g(x-at) & \text{if } x-ct>0 \end{cases}.$$

All that's left to do is to determine f and g. If $t=0$, then the $x-ct>0$ condition applies, so we use the initial conditions for u to determine g.

$$\begin{aligned} u(x,0) &= g(x) = 0 \\ u_t(x,0) &= V - ag'(x) = V \end{aligned}$$

Both are satisfied by setting $g(x-at)=0$.

$$u(x,t) = \begin{cases} f(x-at) & \text{if } x-ct<0 \\ Vt & \text{if } x-ct>0 \end{cases}.$$

Now we turn our attention to finding f. Since $du/dt = V$ in the blue region, u increases linearly in t at the rate V along the characteristic curves $x = at + \xi$. Once the characteristics cross over into the magenta region, u remains constant along them because $du/dt = 0$ there. We require u to be continuous along $x - ct = 0$, the line that separates the blue and magenta regions.

$$f(ct-at) = Vt$$

Let

$$s = ct - at \quad \rightarrow \quad t = -\frac{s}{a-c}.$$

Then

$$f(s) = -\frac{V}{a-c}s.$$

Replacing s with $x-at$, we find that

$$f(x-at) = -\frac{V}{a-c}(x-at).$$

Therefore,

$$u(x,t) = \begin{cases} \dfrac{V}{a-c}(at-x) & \text{if } x-ct<0 \\ Vt & \text{if } x-ct>0 \end{cases}.$$