Exercise 9

(a) Find $u(\frac{2}{3}, 2)$ if $u_{tt} = u_{xx}$ in $0 < x < 1$, $u(x, 0) = x^2(1 - x)$, $u_t(x, 0) = (1 - x)^2$, $u(0, t) = u(1, t) = 0$.

(b) Find $u(\frac{1}{4}, \frac{7}{2})$.

Solution

The method of reflection will be used to solve this problem on a finite interval. Consider the corresponding problem over the whole line, using the extensions for the initial data that are odd with respect to $x = 0$ and $x = 1$ in order to satisfy the homogeneous boundary conditions.

$$\begin{aligned} &v_{tt} = v_{xx} \qquad -\infty < x < \infty,\ t > 0 \\ &v(x, 0) = \phi_{\text{ext}}(x) \\ &v_t(x, 0) = \psi_{\text{ext}}(x) \end{aligned}$$

Here $\phi_{\text{ext}}(x)$ and $\psi_{\text{ext}}(x)$ are defined as

$$\phi_{\text{ext}}(x) = \begin{cases} x^2(1-x) & \text{if } 0 < x < 1 \\ -(-x)^2[1-(-x)] & \text{if } -1 < x < 0 \\ \phi_{\text{ext}}(x+2) \end{cases} \quad \text{and} \quad \psi_{\text{ext}}(x) = \begin{cases} (1-x)^2 & \text{if } 0 < x < 1 \\ -[1-(-x)]^2 & \text{if } -1 < x < 0 \\ \psi_{\text{ext}}(x+2) \end{cases}.$$

The solution for v is given by d'Alembert's formula.

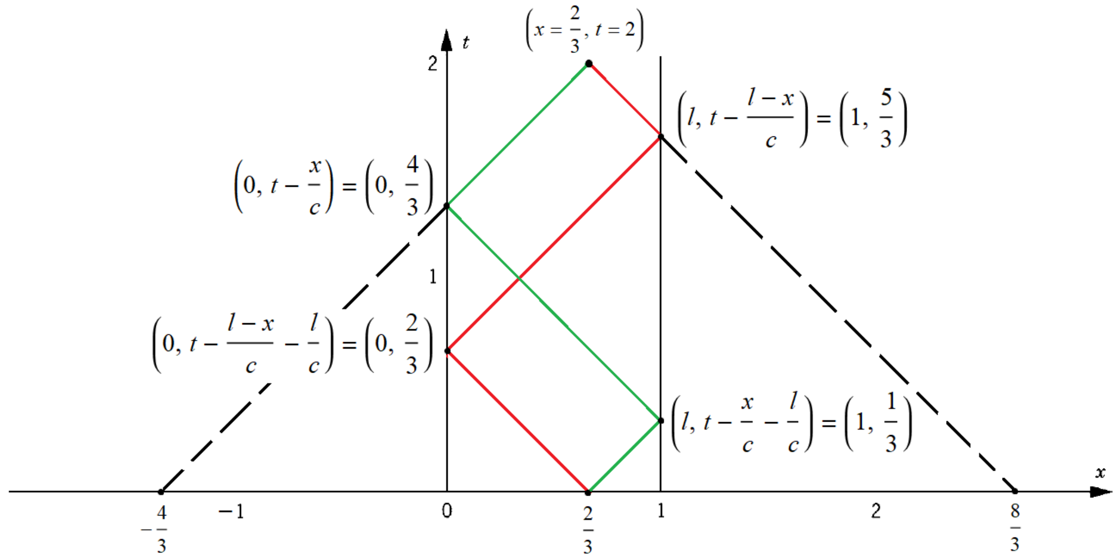
$$v(x, t) = \frac{1}{2}[\phi_{\text{ext}}(x+t) + \phi_{\text{ext}}(x-t)] + \frac{1}{2}\int_{x-t}^{x+t} \psi_{\text{ext}}(s)\, ds$$

u is obtained by restricting this solution to $0 < x < 1$.

$$u(x, t) = \frac{1}{2}[\phi_{\text{ext}}(x+t) + \phi_{\text{ext}}(x-t)] + \frac{1}{2}\int_{x-t}^{x+t} \psi_{\text{ext}}(s)\, ds, \quad 0 < x < 1$$

Part (a)

Our task now is to simplify this formula for the case that $x = 2/3$ and $t = 2$.



There are two reflections on the way to 2/3 along the green path (so two minus signs), and there are two reflections on the way to 2/3 along the red path (so two minus signs).

$$u\left(\frac{2}{3},2\right) = \frac{1}{2}\left[(-1)^2\left(\frac{2}{3}\right)^2\left(1-\frac{2}{3}\right) + (-1)^2\left(\frac{2}{3}\right)^2\left(1-\frac{2}{3}\right)\right]$$
$$+\frac{1}{2}\left[\int_{-\frac{4}{3}}^{-1}[1-(s+2)]^2\,ds + \int_{-1}^{0}[-(1-(-s))^2]\,ds + \int_0^1 (1-s)^2\,ds\right.$$
$$\left.+\int_1^2[-(1-(-s+2))^2]\,ds + \int_2^{\frac{8}{3}}[1-(s-2)]^2\,ds\right]$$

Substitute $r = s+2$ in the first integral, $r = -s$ in the second integral, $r = s$ in the third integral, $r = -s+2$ in the fourth integral, and $r = s-2$ in the fifth integral.

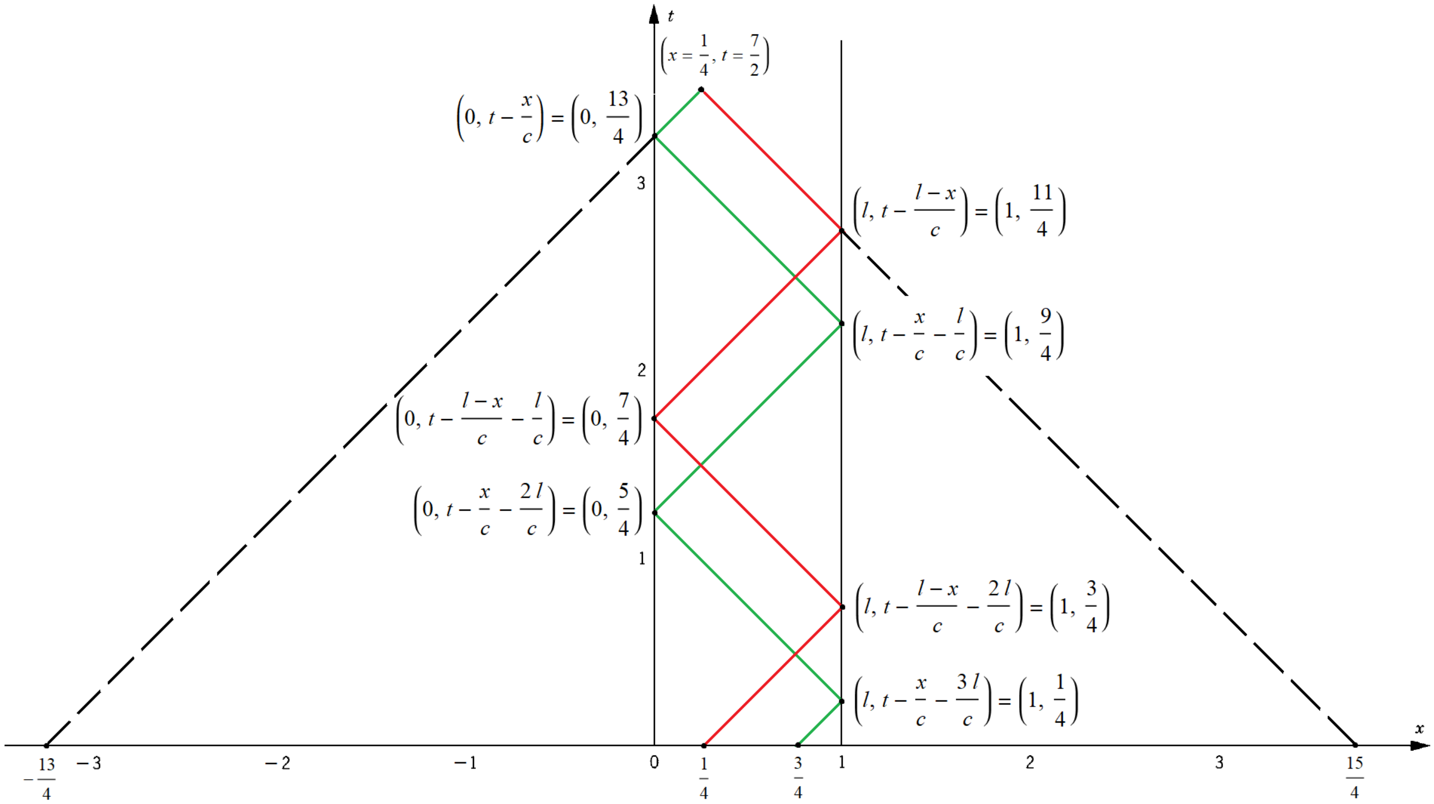
$$u\left(\frac{2}{3},2\right) = \frac{4}{27} + \frac{1}{2}\left[\int_{\frac{2}{3}}^{1}(1-r)^2\,dr + \cancel{\int_1^0 (1-r)^2\,dr} + \cancel{\int_0^1 (1-r)^2\,dr}\right.$$
$$\left.+\int_1^0 (1-r)^2\,dr + \int_0^{\frac{2}{3}}(1-r)^2\,dr\right]$$
$$= \frac{4}{27} + \underbrace{\frac{1}{2}\int_{\frac{2}{3}}^{\frac{2}{3}}(1-r)^2\,dr}_{=\,0}$$

Therefore,

$$u\left(\frac{2}{3},2\right) = \frac{4}{27}.$$

Part (b)

Our task now is to simplify this formula for the case that $x = 1/4$ and $t = 7/2$.



There are four reflections on the way to 3/4 (so four minus signs), and there are three reflections on the way to 1/4 (so three minus signs).

$$u\left(\frac{1}{4},\frac{7}{2}\right) = \frac{1}{2}\left[(-1)^3\left(\frac{1}{4}\right)^2\left(1-\frac{1}{4}\right) + (-1)^4\left(\frac{3}{4}\right)^2\left(1-\frac{3}{4}\right)\right]$$
$$+ \frac{1}{2}\left[\int_{-\frac{13}{4}}^{-3}[1-(s+4)]^2\,ds + \int_{-3}^{-2}[-(1-(-s-2))^2]\,ds + \int_{-2}^{-1}[1-(s+2)]^2\,ds\right.$$
$$+ \int_{-1}^{0}[-(1-(-s))^2]\,ds + \int_{0}^{1}(1-s)^2\,ds + \int_{1}^{2}[-(1-(-s+2))^2]\,ds$$
$$\left. + \int_{2}^{3}[1-(s-2)]^2\,ds + \int_{3}^{\frac{15}{4}}[-(1-(-s+4))^2]\,ds\right]$$

Substitute $r = s + 4$ in the first integral, $r = -s - 2$ in the second integral, $r = s + 2$ in the third integral, $r = -s$ in the fourth integral, $r = s$ in the fifth integral, $r = -s + 2$ in the sixth integral,

$r = s - 2$ in the seventh integral, and $r = -s + 4$ in the eighth integral.

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\begin{aligned}
u\left(\frac{1}{4}, \frac{7}{2}\right) &= \frac{3}{64} + \frac{1}{2}\left[\int_{\frac{3}{4}}^{1}(1-r)^2\,dr + \cancel{\int_{1}^{0}(1-r)^2\,dr} + \cancel{\int_{0}^{1}(1-r)^2\,dr}\right. \\
&\qquad + \bcancel{\int_{1}^{0}(1-r)^2\,dr} + \bcancel{\int_{0}^{1}(1-r)^2\,dr} + \xcancel{\int_{1}^{0}(1-r)^2\,dr} \\
&\qquad \left. + \xcancel{\int_{0}^{1}(1-r)^2\,dr} + \int_{1}^{\frac{1}{4}}(1-r)^2\,dr\right] \\
&= \frac{3}{64} + \frac{1}{2}\int_{\frac{3}{4}}^{\frac{1}{4}}(1-r)^2\,dr \\
&= \frac{3}{64} + \frac{1}{2}\left(-\frac{13}{96}\right)
\end{aligned}
$$

Therefore,

$$
u\left(\frac{1}{4}, \frac{7}{2}\right) = -\frac{1}{48}.
$$