Exercise 1

Solve the Neumann problem for the wave equation on the half-line $0 < x < \infty$.

Solution

The Neumann problem for the wave equation on the half-line is

$$\begin{gathered} v_{tt} - c^2 v_{xx} = 0, \quad 0 < x < \infty,\ -\infty < t < \infty \\ v(x,0) = \phi(x), \quad v_t(x,0) = \psi(x) \\ v_x(0,t) = 0. \end{gathered}$$

Since we're interested in the solution on $0 < x < \infty$, the method of reflection can be applied to solve the PDE. Consider the same problem over the whole line, where the even extensions of the given functions are used in order to satisfy the Neumann boundary condition at $x = 0$.

$$\begin{gathered} u_{tt} - c^2 u_{xx} = 0, \quad -\infty < x < \infty,\ -\infty < t < \infty \\ u(x,0) = \phi_{\text{even}}(x), \quad u_t(x,0) = \psi_{\text{even}}(x), \end{gathered}$$

where

$$\phi_{\text{even}}(x) = \begin{cases} \phi(x) & \text{if } x > 0 \\ \phi(-x) & \text{if } x < 0 \end{cases} \quad \text{and} \quad \psi_{\text{even}}(x) = \begin{cases} \psi(x) & \text{if } x > 0 \\ \psi(-x) & \text{if } x < 0 \end{cases}.$$

The solution for u is given by d'Alembert's formula in section 2.1 on page 36.

$$u(x,t) = \frac{1}{2}[\phi_{\text{even}}(x+ct) + \phi_{\text{even}}(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi_{\text{even}}(s)\, ds$$

The solution for v is then just the restriction of u to $x > 0$.

$$v(x,t) = \frac{1}{2}[\phi_{\text{even}}(x+ct) + \phi_{\text{even}}(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi_{\text{even}}(s)\, ds, \quad x > 0$$

Our task now is to write this formula in terms of the given functions, ϕ and ψ. Note that

$$\phi_{\text{even}}(x+ct) = \begin{cases} \phi(x+ct) & \text{if } x+ct > 0 \\ \phi(-x-ct) & \text{if } x+ct < 0 \end{cases} \quad \text{and} \quad \phi_{\text{even}}(x-ct) = \begin{cases} \phi(x-ct) & \text{if } x-ct > 0 \\ \phi(-x+ct) & \text{if } x-ct < 0 \end{cases},$$

so for every region in the xt-half-plane, we have to test whether $x - ct$ and $x + ct$ are greater than or less than zero. The characteristic curves, $x + ct = 0$ and $x - ct = 0$, are the lines that separate the regions. They are illustrated below in Figure 1.

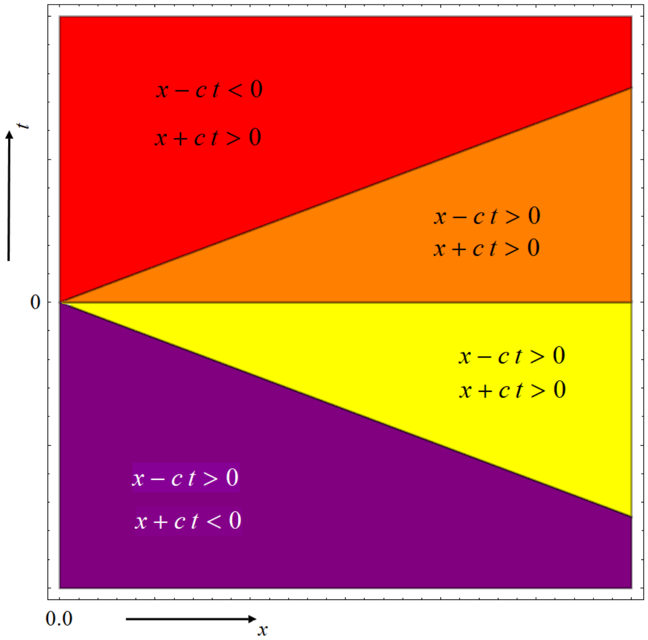

Figure 1: This figure illustrates the regions in the xt-half-plane that come about from using the even extensions of ϕ and ψ. The solution for v has to be considered in each one. The characteristic lines, $x + ct = 0$ and $x - ct = 0$, are the lines that separate the regions. It turns out it was not necessary to separate the orange and yellow ones because $x + ct$ and $x - ct$ are the same in both.

The Red Region

In the red region $x + ct > 0$ and $x - ct < 0$, so the solution for v is

$$\begin{aligned} v(x,t) &= \frac{1}{2}[\phi_{\text{even}}(x+ct) + \phi_{\text{even}}(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi_{\text{even}}(s)\,ds \\ &= \frac{1}{2}[\phi(x+ct) + \phi(-x+ct)] + \frac{1}{2c}\left[\int_{x-ct}^{0} \psi(-s)\,ds + \int_0^{x+ct} \psi(s)\,ds\right]. \end{aligned}$$

Substitute $q = -s$ and $dq = -ds$ in the first integral.

$$\begin{aligned} &= \frac{1}{2}[\phi(x+ct) + \phi(-x+ct)] + \frac{1}{2c}\left[\int_{-x+ct}^{0} \psi(q)(-dq) + \int_0^{x+ct} \psi(s)\,ds\right] \\ &= \frac{1}{2}[\phi(x+ct) + \phi(-x+ct)] + \frac{1}{2c}\left[\int_0^{-x+ct} \psi(q)\,dq + \int_0^{x+ct} \psi(s)\,ds\right] \end{aligned}$$

The Orange & Yellow Regions

In both the orange and yellow regions $x + ct > 0$ and $x - ct > 0$, so the solution for v is

$$\begin{aligned} v(x,t) &= \frac{1}{2}[\phi_{\text{even}}(x+ct) + \phi_{\text{even}}(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi_{\text{even}}(s)\,ds \\ &= \frac{1}{2}[\phi(x+ct) + \phi(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)\,ds. \end{aligned}$$

The Purple Region

In the purple region $x + ct < 0$ and $x - ct > 0$, so the solution for v is

$$\begin{aligned}
v(x,t) &= \frac{1}{2}[\phi_{\text{even}}(x+ct) + \phi_{\text{even}}(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi_{\text{even}}(s)\,ds \\
&= \frac{1}{2}[\phi(-x-ct) + \phi(x-ct)] - \frac{1}{2c}\int_{x+ct}^{x-ct} \psi_{\text{even}}(s)\,ds \\
&= \frac{1}{2}[\phi(-x-ct) + \phi(x-ct)] - \frac{1}{2c}\left[\int_{x+ct}^{0} \psi(-s)\,ds + \int_{0}^{x-ct} \psi(s)\,ds\right].
\end{aligned}$$

Substitute $q = -s$ and $dq = -ds$ in the first integral.

$$\begin{aligned}
&= \frac{1}{2}[\phi(-x-ct) + \phi(x-ct)] - \frac{1}{2c}\left[\int_{-x-ct}^{0} \psi(q)(-dq) + \int_{0}^{x-ct} \psi(s)\,ds\right] \\
&= \frac{1}{2}[\phi(x-ct) + \phi(-x-ct)] - \frac{1}{2c}\left[\int_{0}^{-x-ct} \psi(q)\,dq + \int_{0}^{x-ct} \psi(s)\,ds\right]
\end{aligned}$$

This formula is basically the same as the one for the red region except that t has been switched to $-t$ here, and the sign of $1/2c$ is negative. These two formulas for v can be combined by making use of absolute value signs and the signum (sign) function $\operatorname{sgn} t$, which is -1 if t is negative and 1 if t is positive, like so.

$$v(x,t) = \frac{1}{2}[\phi(x+c|t|) + \phi(-x+c|t|)] + \frac{\operatorname{sgn} t}{2c}\left[\int_{0}^{-x+c|t|} \psi(q)\,dq + \int_{0}^{x+c|t|} \psi(s)\,ds\right], \quad x - c|t| < 0$$

$x - c|t| < 0$ describes the union of the red and purple regions in the xt-half-plane. On the other hand, $x - c|t| > 0$ describes the union of the orange and yellow regions. So then

$$v(x,t) = \begin{cases} \frac{1}{2}[\phi(x+c|t|) + \phi(-x+c|t|)] + \frac{\operatorname{sgn} t}{2c}\left[\int_{0}^{-x+c|t|} \psi(q)\,dq + \int_{0}^{x+c|t|} \psi(s)\,ds\right] & \text{if } x - c|t| < 0 \\ \frac{1}{2}[\phi(x+ct) + \phi(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)\,ds & \text{if } x - c|t| > 0 \end{cases}.$$

The solution will be checked now to see if it satisfies the Neumann boundary condition. If $x = 0$, then the $x - c|t| < 0$ condition applies, so the magenta formula has to satisfy $v_x(0,t) = 0$.

$$v_x(x,t) = \frac{1}{2}[\phi'(x+c|t|) - \phi'(-x+c|t|)] + \frac{\operatorname{sgn} t}{2c}\left[\frac{d}{dx}\int_{0}^{-x+c|t|} \psi(q)\,dq + \frac{d}{dx}\int_{0}^{x+c|t|} \psi(s)\,ds\right]$$

Apply the Leibniz rule to differentiate the integrals.

$$v_x(x,t) = \frac{1}{2}[\phi'(x+c|t|) - \phi'(-x+c|t|)] + \frac{\operatorname{sgn} t}{2c}\left[\psi(-x+c|t|)\cdot(-1) + \psi(x+c|t|)\cdot 1\right]$$

Plugging in $x = 0$, we see that the Neumann boundary condition is satisfied.

$$v_x(0,t) = \frac{1}{2}[\cancel{\phi'(c|t|)} - \cancel{\phi'(c|t|)}] + \frac{\operatorname{sgn} t}{2c}[-\cancel{\psi(c|t|)} + \cancel{\psi(c|t|)}] = 0$$

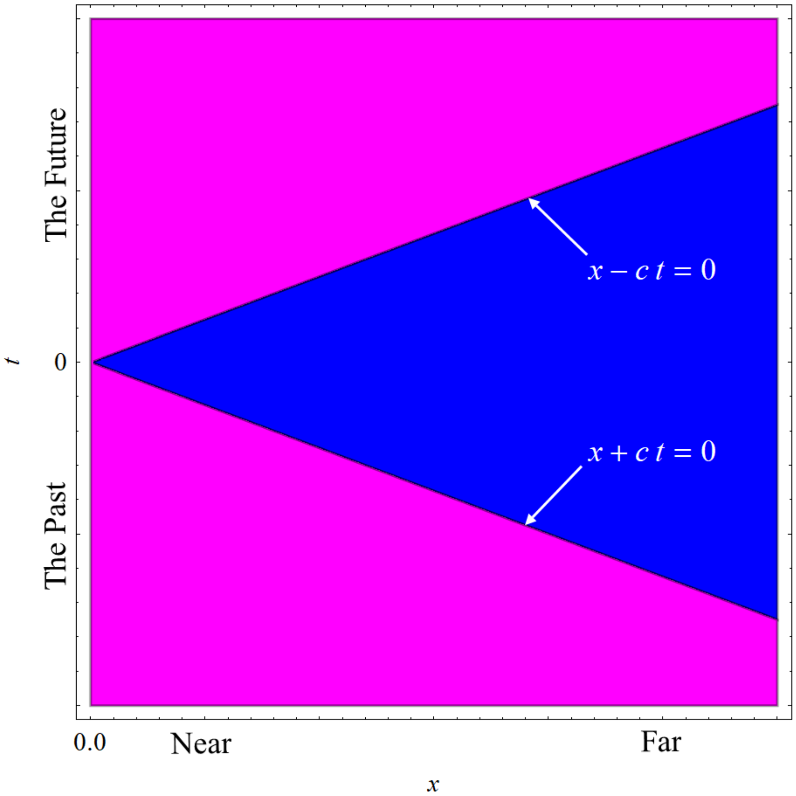

Figure 2: This figure shows the regions in the xt-half-plane where the colored formulas are valid. In magenta, the region corresponding to $x - c|t| < 0$, the first formula applies, and in blue, the region corresponding to $x - c|t| > 0$, the second formula applies. The characteristic lines, $x + ct = 0$ and $x - ct = 0$, separate the regions.

Note that any constant C can be added to the magenta solution, and the result will still satisfy the wave equation and the boundary condition.

$$v(x,t) = \frac{1}{2}[\phi(x+c|t|) + \phi(-x+c|t|)] + \frac{\operatorname{sgn} t}{2c}\left[\int_0^{-x+c|t|} \psi(q)\, dq + \int_0^{x+c|t|} \psi(s)\, ds\right] + C, \quad x - c|t| < 0$$

C can be conveniently chosen so that all the constants in the solution sum to zero.

$$\frac{\operatorname{sgn} t}{2c}\left[\int_0 \psi(q)\, dq + \int_0 \psi(s)\, ds\right] + C = 0, \quad x - c|t| < 0$$

Therefore,

$$v(x,t) = \begin{cases} \frac{1}{2}[\phi(x+c|t|) + \phi(-x+c|t|)] + \frac{\operatorname{sgn} t}{2c}\left[\int^{-x+c|t|} \psi(q)\, dq + \int^{x+c|t|} \psi(s)\, ds\right] & \text{if } x - c|t| < 0 \\ \frac{1}{2}[\phi(x+ct) + \phi(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)\, ds & \text{if } x - c|t| > 0 \end{cases}.$$