Exercise 2

The longitudinal vibrations of a semi-infinite flexible rod satisfy the wave equation $u_{tt} = c^2 u_{xx}$ for $x > 0$. Assume that the end $x = 0$ is free ($u_x = 0$); it is initially at rest but has a constant initial velocity V for $a < x < 2a$ and has zero initial velocity elsewhere. Plot u versus x at the times $t = 0$, a/c, $3a/2c$, $2a/c$, and $3a/c$.

Solution

The governing PDE for the longitudinal vibrations and its associated initial and boundary conditions are

$$u_{tt} = c^2 u_{xx}, \quad 0 < x < \infty,\ 0 < t < \infty$$
$$u(x,0) = 0, \quad u_t(x,0) = \psi(x) = \begin{cases} V & a < x < 2a \\ 0 & \text{elsewhere} \end{cases}$$
$$u_x(0,t) = 0.$$

Since we're interested in the solution on $0 < x < \infty$, the method of reflection can be applied to solve the equation. Consider the same problem over the whole line, where the even extension of the given function is used in order to satisfy the Neumann boundary condition at $x = 0$.

$$v_{tt} = c^2 v_{xx}, \quad -\infty < x < \infty,\ 0 < t < \infty$$
$$v(x,0) = 0, \quad v_t(x,0) = \psi_{\text{even}}(x) = \begin{cases} V & -2a < x < -a \\ V & a < x < 2a \\ 0 & \text{elsewhere} \end{cases}$$

The solution for v is given by d'Alembert's formula in section 2.1 on page 36.

$$v(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} \psi_{\text{even}}(s)\,ds$$

The solution for u is then just the restriction of v to $x > 0$.

$$u(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} \psi_{\text{even}}(s)\,ds, \quad x > 0 \tag{1}$$

Our task now is to graph this function for the five times of interest.

<u>Plot of u versus x at time $t = 0$</u>

If $t = 0$, then equation (1) reduces to

$$u(x,0) = \frac{1}{2c}\int_{x}^{x} \psi_{\text{even}}(s)\,ds = 0,$$

which is the given initial condition.

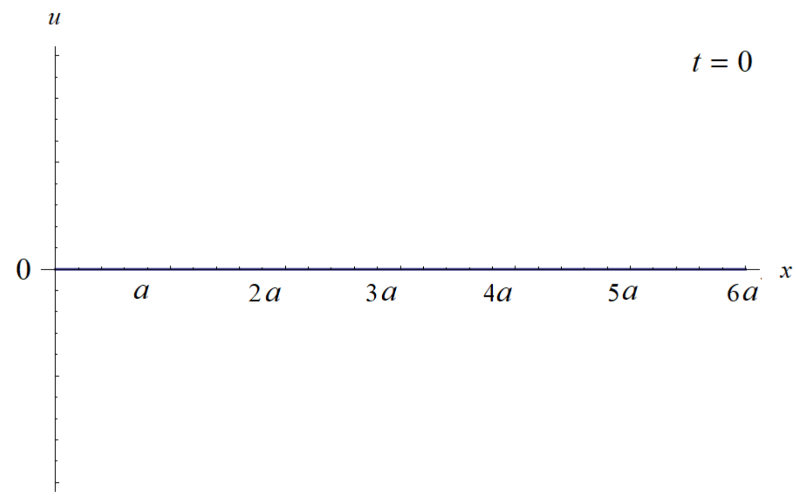

Figure 1: This figure shows the plot of u versus x at time $t = 0$.

Plot of u versus x at time $t = a/c$

If $t = a/c$, then the limits of integration in equation (1) become

$$x + ct = x + c\left(\frac{a}{c}\right) = x + a$$
$$x - ct = x - c\left(\frac{a}{c}\right) = x - a.$$

So the solution for u is

$$u\left(x, \frac{a}{c}\right) = \frac{1}{2c}\int_{x-a}^{x+a} \psi_{\text{even}}(s)\, ds.$$

The interval of integration is essentially a rod of length $2a$: $x - a$ can be thought of as its left end, x can be thought of as its midpoint, and $x + a$ can be thought of as its right end. At $x = 0$ the rod lies between $-a$ and a. In this interval ψ_{even} is zero, so the integral is zero.

$$u\left(0, \frac{a}{c}\right) = \frac{1}{2c}\int_{-a}^{a} \psi_{\text{even}}(s)\, ds = 0$$

Consider now the case when $0 < x < a$. The rod's right end is inside the interval $(a, 2a)$ to some depth.

$$u\left(x, \frac{a}{c}\right) = \frac{1}{2c}\left(\int_{x-a}^{a} 0\, ds + \int_{a}^{x+a} V\, ds\right) = \frac{V}{2c}x, \quad 0 < x < a$$

Consider now the case when $a < x < 2a$. The right end of the rod extends past $(a, 2a)$, and the left end of the rod approaches $(a, 2a)$.

$$u\left(x, \frac{a}{c}\right) = \frac{1}{2c}\left(\int_{x-a}^{a} 0\, ds + \int_{a}^{2a} V\, ds + \int_{2a}^{x+a} 0\, ds\right) = \frac{aV}{2c}, \quad a < x < 2a$$

Consider now the case when $2a < x < 3a$. The left end of the rod is inside $(a, 2a)$ to some depth. The right end of the rod keeps moving to the right.

$$u\left(x, \frac{a}{c}\right) = \frac{1}{2c}\left(\int_{x-a}^{2a} V\, ds + \int_{2a}^{x+a} 0\, ds\right) = \frac{V}{2c}(3a - x), \quad 2a < x < 3a$$

Consider now the case when $x > 3a$. The left end of the rod is out past $(a, 2a)$. Because ψ_{even} is zero everywhere outside of $(-2a, -a)$ and $(a, 2a)$, the integral is zero.

$$u\left(x, \frac{a}{c}\right) = \frac{1}{2c}\int_{x-a}^{x+a} 0\, ds = 0, \quad x > 3a$$

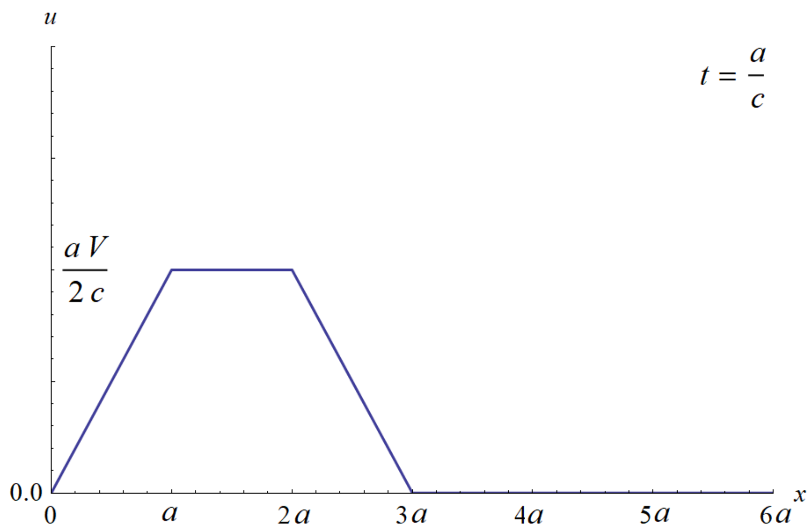

Figure 2: This figure shows the plot of u versus x at time $t = a/c$.

Plot of u versus x at time $t = 3a/2c$

If $t = 3a/2c$, then the limits of integration in equation (1) become

$$\begin{aligned} x + ct &= x + c\left(\frac{3a}{2c}\right) = x + \frac{3a}{2} \\ x - ct &= x - c\left(\frac{3a}{2c}\right) = x - \frac{3a}{2}. \end{aligned}$$

So the solution for u is

$$u\left(x, \frac{3a}{2c}\right) = \frac{1}{2c}\int_{x-\frac{3a}{2}}^{x+\frac{3a}{2}} \psi_{\text{even}}(s)\, ds.$$

The interval of integration is essentially a rod of length $3a$: $x - 3a/2$ can be thought of as its left end, x can be thought of as its midpoint, and $x + 3a/2$ can be thought of as its right end. At $x = 0$ the rod lies between $-3a/2$ and $3a/2$. The left end of the rod is halfway through the interval $(-2a, -a)$, and the right end of the rod is halfway through the interval $(a, 2a)$.

$$u\left(0, \frac{3a}{2c}\right) = \frac{1}{2c}\int_{-\frac{3a}{2}}^{\frac{3a}{2}} \psi_{\text{even}}(s)\, ds = \frac{1}{2c}\left(\int_{-\frac{3a}{2}}^{-a} V\, ds + \int_{-a}^{a} 0\, ds + \int_{a}^{\frac{3a}{2}} V\, ds\right) = \frac{aV}{2c}.$$

Consider now the case when $0 < x < a/2$. The right end of the rod goes deeper into the interval $(a, 2a)$, and the left end begins to pull out of the interval $(-2a, -a)$.

$$u\left(x, \frac{3a}{2c}\right) = \frac{1}{2c}\left(\int_{x-\frac{3a}{2}}^{-a} V\, ds + \int_{-a}^{a} 0\, ds + \int_{a}^{x+\frac{3a}{2}} V\, ds\right) = \frac{aV}{2c}, \quad 0 < x < \frac{a}{2}$$

Consider now the case when $a/2 < x < 5a/2$. No part of the rod lies in $(-2a, -a)$ anymore. The rod completely covers $(a, 2a)$, and the left end of the rod approaches $(a, 2a)$.

$$u\left(x, \frac{3a}{2c}\right) = \frac{1}{2c}\left(\int_{x-\frac{3a}{2}}^{a} 0\, ds + \int_{a}^{2a} V\, ds + \int_{2a}^{x+\frac{3a}{2}} 0\, ds\right) = \frac{aV}{2c}, \quad \frac{a}{2} < x < \frac{5a}{2}$$

Consider now the case when $5a/2 < x < 7a/2$. The left end of the rod is inside $(a, 2a)$ to some depth, and the right end of the rod keeps moving to the right.

$$u\left(x, \frac{3a}{2c}\right) = \frac{1}{2c}\left(\int_{x-\frac{3a}{2}}^{2a} V\, ds + \int_{2a}^{x+\frac{3a}{2}} 0\, ds\right) = \frac{V}{2c}\left(\frac{7a}{2} - x\right), \quad \frac{5a}{2} < x < \frac{7a}{2}$$

Consider now the case when $x > 7a/2$. The left end of the rod is out past $(a, 2a)$. Because ψ_{even} is zero everywhere outside of $(-2a, -a)$ and $(a, 2a)$, the integral is zero.

$$u\left(x, \frac{3a}{2c}\right) = \frac{1}{2c}\int_{x-\frac{3a}{2}}^{x+\frac{3a}{2}} 0\, ds = 0, \quad x > \frac{7a}{2}$$

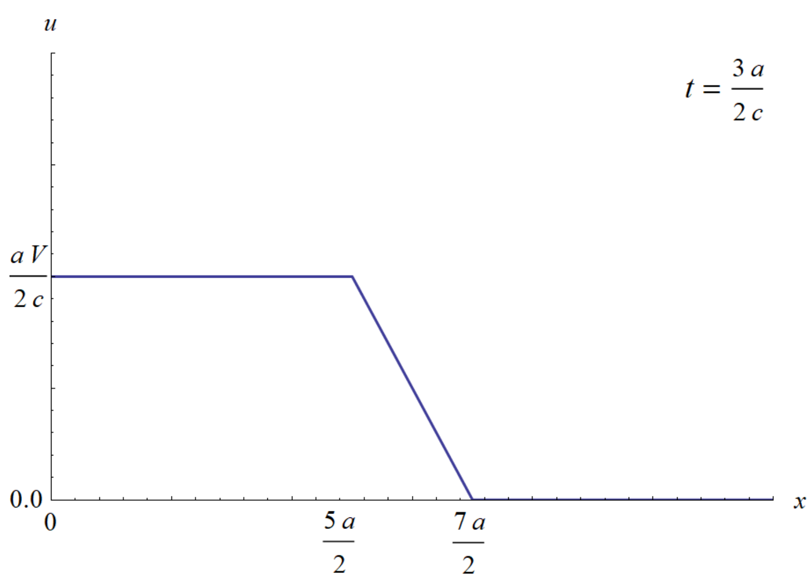

Figure 3: This figure shows the plot of u versus x at time $t = 3a/2c$.

Plot of u versus x at time $t = 2a/c$

If $t = 2a/c$, then the limits of integration in equation (1) become

$$x + ct = x + c\left(\frac{2a}{c}\right) = x + 2a$$
$$x - ct = x - c\left(\frac{2a}{c}\right) = x - 2a.$$

So the solution for u is

$$u\left(x, \frac{2a}{c}\right) = \frac{1}{2c}\int_{x-2a}^{x+2a} \psi_{\text{even}}(s)\, ds.$$

The interval of integration is essentially a rod of length $4a$: $x-2a$ can be thought of as its left end, x can be thought of as its midpoint, and $x+2a$ can be thought of as its right end. At $x=0$ the rod lies between $-2a$ and $2a$, which completely covers both $(-2a,-a)$ and $(a,2a)$.

$$u\left(0,\frac{2a}{c}\right)=\frac{1}{2c}\int_{-2a}^{2a}\psi_{\text{even}}(s)\,ds=\frac{1}{2c}\left(\int_{-2a}^{-a}V\,ds+\int_{-a}^{a}0\,ds+\int_{a}^{2a}V\,ds\right)=\frac{aV}{c}$$

Consider now the case when $0<x<a$. The left end of the rod starts to pull out of $(-2a,-a)$, and the right end of the rod extends past $(a,2a)$. The rod covers the whole interval $(a,2a)$.

$$u\left(x,\frac{2a}{c}\right)=\frac{1}{2c}\left(\int_{x-2a}^{-a}V\,ds+\int_{-a}^{a}0\,ds+\int_{a}^{2a}V\,ds+\int_{2a}^{x+2a}0\,ds\right)=\frac{V}{2c}(2a-x),\quad 0<x<a$$

Consider now the case when $a<x<3a$. The left end of the rod is out of $(-2a,-a)$ and approaches $(a,2a)$. The rod covers the whole interval $(a,2a)$.

$$u\left(x,\frac{2a}{c}\right)=\frac{1}{2c}\left(\int_{x-2a}^{a}0\,ds+\int_{a}^{2a}V\,ds+\int_{2a}^{x+2a}0\,ds\right)=\frac{aV}{2c},\quad a<x<3a$$

Consider now the case when $3a<x<4a$. The left end of the rod is in $(a,2a)$ to some depth. The right end of the rod continues to move to the right.

$$u\left(x,\frac{2a}{c}\right)=\frac{1}{2c}\left(\int_{x-2a}^{2a}V\,ds+\int_{2a}^{x+2a}0\,ds\right)=\frac{V}{2c}(4a-x),\quad 3a<x<4a$$

Consider now the case when $x>4a$. The left end of the rod is out past $(a,2a)$. Since ψ_{even} is zero everywhere outside of $(-2a,-a)$ and $(a,2a)$, the integral is zero.

$$u\left(x,\frac{2a}{c}\right)=\frac{1}{2c}\int_{x-2a}^{x+2a}0\,ds=0,\quad x>4a$$

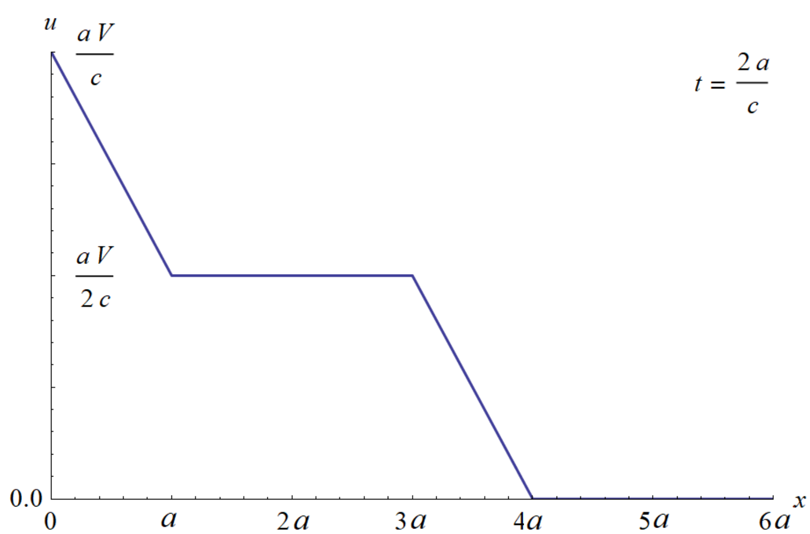

Figure 4: This figure shows the plot of u versus x at time $t=2a/c$.

Plot of u versus x at time $t = 3a/c$

If $t = 3a/c$, then the limits of integration in equation (1) become

$$x + ct = x + c\left(\frac{3a}{c}\right) = x + 3a$$
$$x - ct = x - c\left(\frac{3a}{c}\right) = x - 3a.$$

So the solution for u is

$$u\left(x, \frac{3a}{c}\right) = \frac{1}{2c}\int_{x-3a}^{x+3a} \psi_{\text{even}}(s)\,ds.$$

The interval of integration is essentially a rod of length $6a$: $x - 3a$ can be thought of as its left end, x can be thought of as its midpoint, and $x + 3a$ can be thought of as its right end. At $x = 0$ the rod lies between $-3a$ and $3a$, which covers both intervals, $(-2a, -a)$ and $(a, 2a)$.

$$u\left(0, \frac{3a}{c}\right) = \frac{1}{2c}\int_{-3a}^{3a} \psi_{\text{even}}(s)\,ds = \frac{1}{2c}\left(\int_{-3a}^{-2a} 0\,ds + \int_{-2a}^{-a} V\,ds + \int_{-a}^{a} 0\,ds + \int_{a}^{2a} V\,ds + \int_{2a}^{3a} 0\,ds\right) = \frac{aV}{c}$$

Consider now the case when $0 < x < a$. The left end of the rod approaches $(-2a, -a)$, and the right end of the rod extends past the right of $(a, 2a)$. The rod still covers both intervals.

$$u\left(x, \frac{3a}{c}\right) = \frac{1}{2c}\left(\int_{x-3a}^{-2a} 0\,ds + \int_{-2a}^{-a} V\,ds + \int_{-a}^{a} 0\,ds + \int_{a}^{2a} V\,ds + \int_{2a}^{x+3a} 0\,ds\right) = \frac{aV}{c}, \quad 0 < x < a$$

Consider now the case when $a < x < 2a$. The left end of the rod is inside $(-2a, -a)$ to some depth, and the right end of the rod extends past the right of $(a, 2a)$. The rod still covers $(a, 2a)$.

$$u\left(x, \frac{3a}{c}\right) = \frac{1}{2c}\left(\int_{x-3a}^{-a} V\,ds + \int_{-a}^{a} 0\,ds + \int_{a}^{2a} V\,ds + \int_{2a}^{x+3a} 0\,ds\right) = \frac{V}{2c}(3a - x), \quad a < x < 2a$$

Consider now the case when $2a < x < 4a$. The left end of the rod is outside past $(-2a, -a)$ and approaches $(a, 2a)$. The rod still covers $(a, 2a)$.

$$u\left(x, \frac{3a}{c}\right) = \frac{1}{2c}\left(\int_{x-3a}^{a} 0\,ds + \int_{a}^{2a} V\,ds + \int_{2a}^{x+3a} 0\,ds\right) = \frac{aV}{2c}, \quad 2a < x < 4a$$

Consider now the case when $4a < x < 5a$. The left end of the rod is inside $(a, 2a)$ to some depth, and the right end of the rod continues to move to the right.

$$u\left(x, \frac{3a}{c}\right) = \frac{1}{2c}\left(\int_{x-3a}^{2a} V\,ds + \int_{2a}^{x+3a} 0\,ds\right) = \frac{V}{2c}(5a - x), \quad 4a < x < 5a$$

Consider now the case when $x > 5a$. The left end of the rod is past $(a, 2a)$. Since ψ_{even} is zero everywhere outside of $(-2a, -a)$ and $(a, 2a)$, the integral is zero.

$$u\left(x, \frac{3a}{c}\right) = \frac{1}{2c}\int_{x-3a}^{x+3a} 0\,ds = 0, \quad x > 5a$$

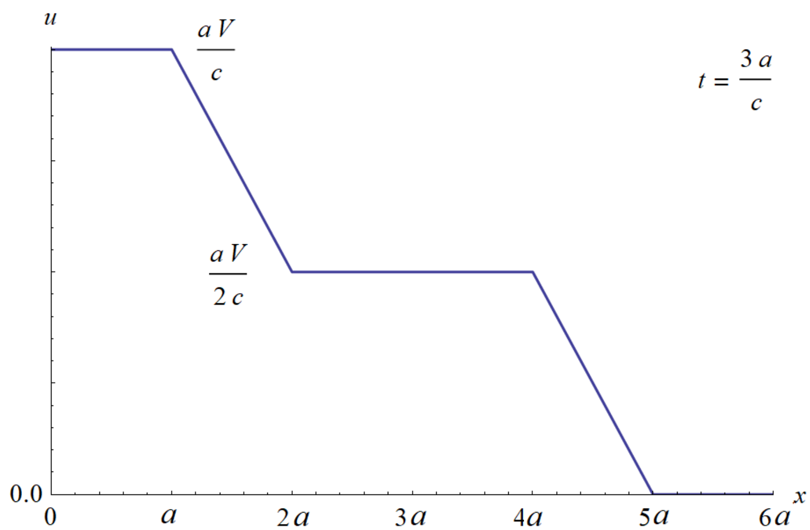

Figure 5: This figure shows the plot of u versus x at time $t = 3a/c$.