# Exercise 3

A wave $f(x+ct)$ travels along a semi-infinite string $(0 < x < \infty)$ for $t < 0$. Find the vibrations $u(x,t)$ of the string for $t > 0$ if the end $x = 0$ is fixed.

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**Solution**

The governing equation of motion for a homogeneous string is the wave equation. If the end of the string at $x = 0$ is fixed, then there is a Dirichlet boundary condition at $x = 0$. The two initial conditions are obtained from the pre-existing wave $f(x+ct)$—one by setting $t = 0$ and the second by differentiating with respect to $t$ and then setting $t = 0$. Consequently, the initial boundary value problem to solve is

$$\begin{gathered} u_{tt} = c^2 u_{xx}, \quad 0 < x < \infty,\ t > 0 \\ u(x,0) = f(x) \quad u_t(x,0) = cf'(x) \\ u(0,t) = 0. \end{gathered}$$

Since we're interested in the solution on $0 < x < \infty$, the method of reflection can be applied to solve the PDE. Consider the same problem over the whole line, where the odd extensions of the given functions are used in order to satisfy the Dirichlet boundary condition at $x = 0$.

$$\begin{gathered} v_{tt} = c^2 v_{xx}, \quad -\infty < x < \infty,\ t > 0 \\ v(x,0) = f_{\text{odd}}(x), \quad v_t(x,0) = cf'_{\text{odd}}(x), \end{gathered}$$

where

$$f_{\text{odd}}(x) = \begin{cases} f(x) & \text{if } x > 0 \\ -f(-x) & \text{if } x < 0 \end{cases} \quad \text{and} \quad cf'_{\text{odd}}(x) = \begin{cases} cf'(x) & \text{if } x > 0 \\ -cf'(-x) & \text{if } x < 0 \end{cases}.$$

The solution for $v$ is given by d'Alembert's formula in section 2.1 on page 36.

$$v(x,t) = \frac{1}{2}[f_{\text{odd}}(x+ct) + f_{\text{odd}}(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} cf'_{\text{odd}}(s)\,ds$$

The solution for $u$ is then just the restriction of $v$ to $x > 0$.

$$u(x,t) = \frac{1}{2}[f_{\text{odd}}(x+ct) + f_{\text{odd}}(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} cf'_{\text{odd}}(s)\,ds, \quad x > 0$$

Our task now is to write this formula in terms of the given function $f$. Note that

$$f_{\text{odd}}(x+ct) = \begin{cases} f(x+ct) & \text{if } x+ct > 0 \\ -f(-x-ct) & \text{if } x+ct < 0 \end{cases} \quad \text{and} \quad f_{\text{odd}}(x-ct) = \begin{cases} f(x-ct) & \text{if } x-ct > 0 \\ -f(-x+ct) & \text{if } x-ct < 0 \end{cases},$$

so for every region in the $xt$-quarter-plane, we have to test whether $x - ct$ and $x + ct$ are greater than or less than zero. The characteristic curve $x - ct = 0$ is the line that separates the regions. They are illustrated below in Figure 1.

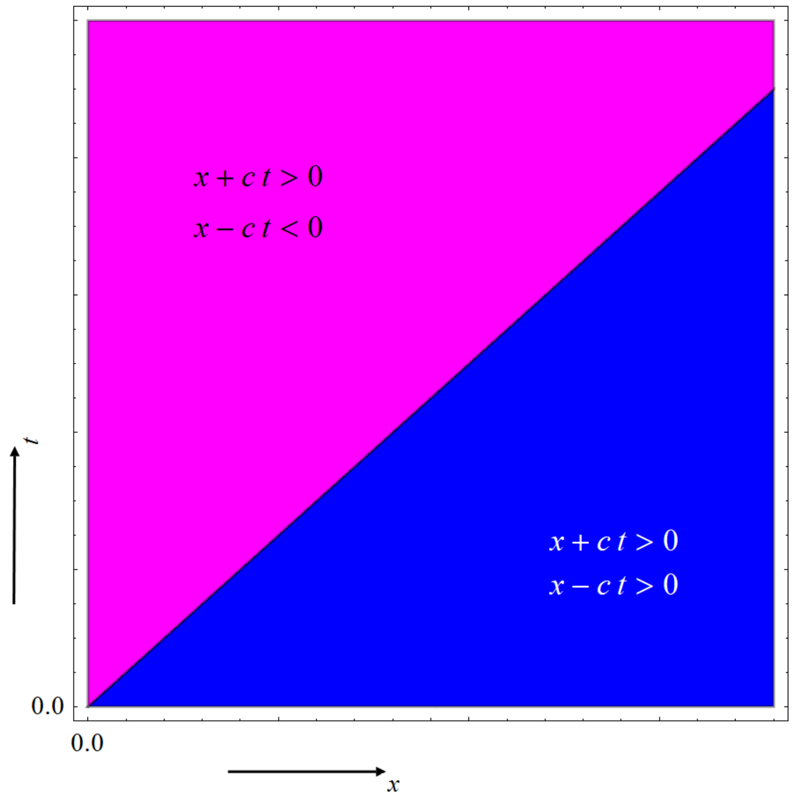


Figure 1: This figure illustrates the regions in the $xt$-quarter-plane that come about from using the odd extensions of $f$ and $cf'$. The solution for $u$ has to be considered in each one. The characteristic line $x - ct = 0$ is the line that separates the regions.

## The Magenta Region

In the magenta region $x + ct > 0$ and $x - ct < 0$, so the solution for $u$ is

$$\begin{aligned} u(x,t) &= \frac{1}{2}[f_{\text{odd}}(x+ct) + f_{\text{odd}}(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} cf'_{\text{odd}}(s)\,ds \\ &= \frac{1}{2}[f(x+ct) - f(-x+ct)] + \frac{1}{2c}\left\{\int_{x-ct}^{0}[-cf'(-s)]\,ds + \int_0^{x+ct} cf'(s)\,ds\right\}. \end{aligned}$$

Cancel $c$ and make the substitution $q = -s$ and $dq = -ds$ in the first integral.

$$\begin{aligned} &= \frac{1}{2}[f(x+ct) - f(-x+ct)] + \frac{1}{2}\left[\int_{-x+ct}^{0} f'(q)\,dq + \int_0^{x+ct} f'(s)\,ds\right] \\ &= \frac{1}{2}[f(x+ct) - f(-x+ct)] + \frac{1}{2}\left[f(q)\Big|_{-x+ct}^{0} + f(s)\Big|_0^{x+ct}\right] \\ &= \frac{1}{2}[f(x+ct) - f(-x+ct)] + \frac{1}{2}\left[\cancel{f(0)} - f(-x+ct) + f(x+ct) - \cancel{f(0)}\right] \\ &= f(x+ct) - f(-x+ct) \end{aligned}$$

## The Blue Region

In the blue region $x + ct > 0$ and $x - ct > 0$, so the solution for $u$ is

$$\begin{aligned} u(x,t) &= \frac{1}{2}[f_{\text{odd}}(x+ct) + f_{\text{odd}}(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} c f'_{\text{odd}}(s)\,ds \\ &= \frac{1}{2}[f(x+ct) + f(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} c f'(s)\,ds \\ &= \frac{1}{2}[f(x+ct) + f(x-ct)] + \frac{1}{2} f(s)\Big|_{x-ct}^{x+ct} \\ &= \frac{1}{2}[f(x+ct) + \cancel{f(x-ct)}] + \frac{1}{2}[f(x+ct) - \cancel{f(x-ct)}] \\ &= f(x+ct). \end{aligned}$$

Therefore, for $t > 0$,

$$u(x,t) = \begin{cases} f(x+ct) - f(-x+ct) & \text{if } x - ct < 0 \\ f(x+ct) & \text{if } x - ct > 0 \end{cases}.$$

Note that if $x = 0$, then the $x - ct < 0$ condition applies, and $u(0,t) = f(ct) - f(ct) = 0$. The Dirichlet boundary condition is satisfied.

## Physical Significance of the Solution

To understand the signficance of the formula for $u$, it's best to consider an example. Suppose the pre-existing waveform is a Gaussian pulse

$$f(s) = e^{-5(s-1)^2}$$

with speed $c = 1$ so that

$$u(x,t) = \begin{cases} e^{-5(x+t-1)^2} - e^{-5(-x+t-1)^2} & \text{if } x - t < 0 \\ e^{-5(x+t-1)^2} & \text{if } x - t > 0 \\ e^{-5(x+t-1)^2} & \text{if } t < 0 \end{cases}.$$

Graphing $u$ versus $x$ for various times of $t$ will give us insight into the physics of a string with a fixed end.

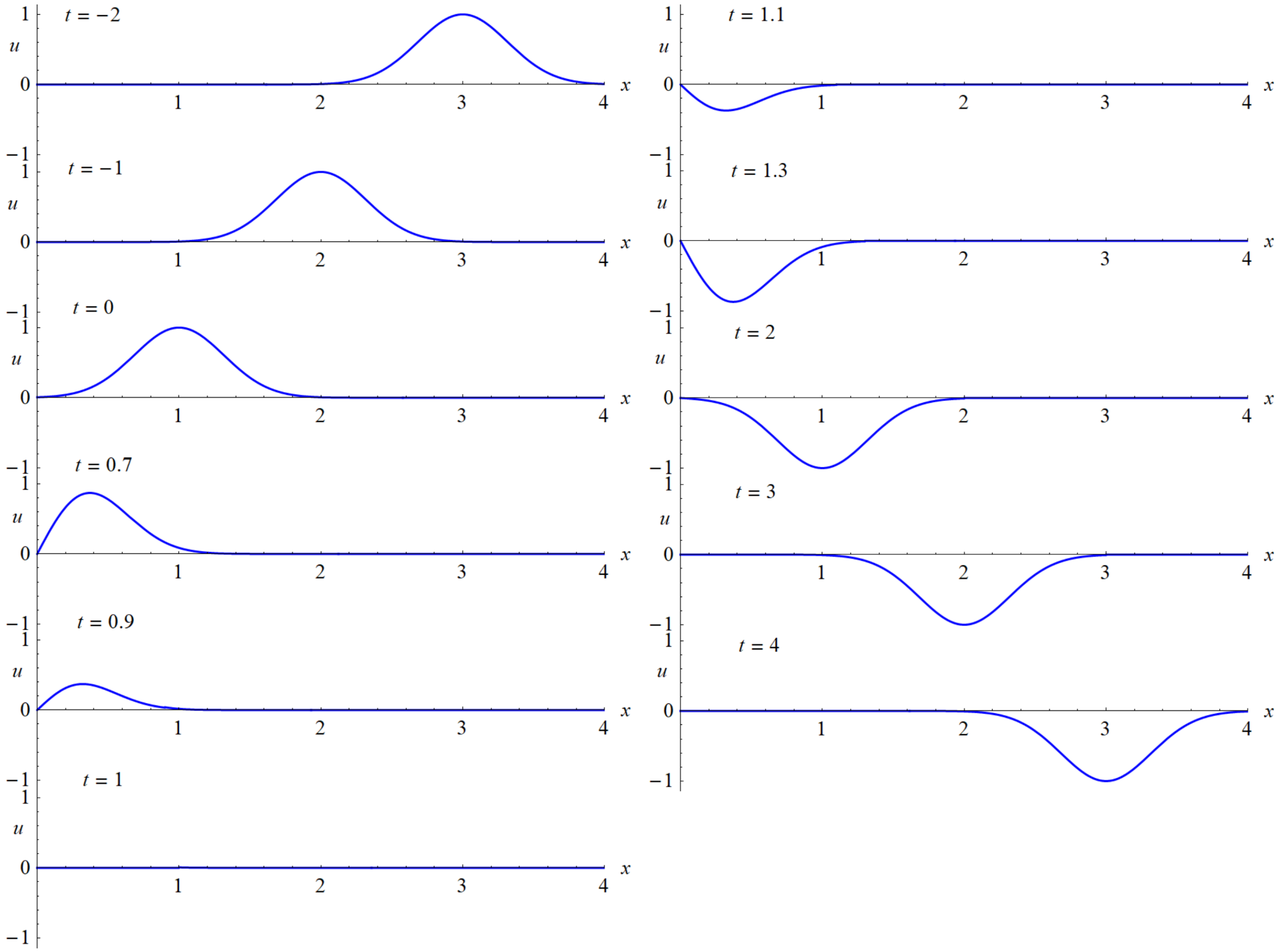


Figure 2: This figure illustrates the time evolution of the Gaussian pulse from 2 seconds into the past to 4 seconds into the future. This can be interpreted physically as the motion of a homogeneous elastic string fixed at the left end. The propagating wave inverts once it reaches the boundary and travels back with the same speed to where it came from.

The solution for $u$ on the whole line is essentially the sum of two waves, one Gaussian pulse travelling from right to left in the upper half of the $xu$-plane and one Gaussian pulse travelling from left to right in the lower half of the $xu$-plane. Once the tails of the curves meet at $t = 0$ and $x = 0$, the waves begin to superimpose and interfere destructively with one another. At $t = 1$ the waves are exactly on top of each other and effectively cancel out. The wave we see for $t > 2$ is the wave that came from the left of $x = 0$ during $t < 0$.