Exercise 6

Derive the formula for the inhomogeneous wave equation in yet another way.

(a) Write it as the system

$$u_t + cu_x = v, \quad v_t - cv_x = f.$$

(b) Solve the first equation for u in terms of v as

$$u(x,t) = \int_0^t v(x - ct + cs, s)\,ds.$$

(c) Similarly, solve the second equation for v in terms of f.

(d) Substitute part (c) into part (b) and write as an iterated integral.

Solution

The PDE we have to solve is

$$u_{tt} = c^2 u_{xx} + f(x,t)$$

over the whole line. There are two initial conditions,

$$u(x,0) = \phi(x) \quad \text{and} \quad u_t(x,0) = \psi(x).$$

Part (a)

Bring $c^2 u_{xx}$ to the other side.

$$u_{tt} - c^2 u_{xx} = f(x,t)$$

Write the left side as an operator acting on u.

$$(\partial_t^2 - c^2 \partial_x^2)u = f(x,t)$$

The operator is a difference of squares, so it can be factored.

$$(\partial_t - c\partial_x)(\partial_t + c\partial_x)u = f(x,t)$$

Let

$$v = (\partial_t + c\partial_x)u$$

so that the PDE becomes

$$(\partial_t - c\partial_x)v = f(x,t).$$

The second-order PDE we started with has thus been reduced to the following system of first-order PDEs that can be solved with the method of characteristics.

$$u_t + cu_x = v \tag{1}$$

$$v_t - cv_x = f(x,t) \tag{2}$$

Part (b)

For a function of two variables $z = z(x, t)$, its differential is defined as

$$dz = \frac{\partial z}{\partial t}\,dt + \frac{\partial z}{\partial x}\,dx.$$

If we divide both sides by dt, then we get the relationship between the ordinary derivative of ϕ and its partial derivatives.

$$\frac{dz}{dt} = \frac{\partial z}{\partial t} + \frac{\partial z}{\partial x}\frac{dx}{dt} \tag{3}$$

Comparing this with equation (1), we see that along the curves in the xt-plane that satisfy

$$\frac{dx}{dt} = c, \tag{4}$$

the PDE for $u(x, t)$ reduces to an ODE.

$$\frac{du}{dt} = v(x, t) \tag{5}$$

Because c is a constant, equation (4) can be solved by integrating both sides with respect to t.

$$x = ct + \xi, \tag{6}$$

where ξ is a characteristic coordinate. Substitute this expression for x into equation (5) to obtain an ODE that only involves t (ξ is regarded as a constant).

$$\frac{du}{dt} = v(ct + \xi, t)$$

Integrate both sides with respect to t.

$$u(\xi, t) = \int_0^t v(cr + \xi, r)\,dr + g(\xi),$$

where g is an arbitrary function of ξ. The lower limit of integration is arbitrary and has been set equal to 0. In order to change back to the original variable x, solve equation (6) for ξ.

$$x = ct + \xi \quad \rightarrow \quad \xi = x - ct$$

Therefore,

$$u(x, t) = \int_0^t v(cr + x - ct, r)\,dr + g(x - ct).$$

Part (c)

Comparing equation (3) with equation (2), we see that along the curves in the xt-plane that satisfy

$$\frac{dx}{dt} = -c, \tag{7}$$

the PDE for $v(x, t)$ reduces to an ODE.

$$\frac{dv}{dt} = f(x, t) \tag{8}$$

Because c is a constant, equation (7) can be solved by integrating both sides with respect to t.

$$x = -ct + \eta, \tag{9}$$

where η is a characteristic coordinate. Substitute this expression for x into equation (8) to obtain an ODE that only involves t (η is regarded as a constant).

$$\frac{dv}{dt} = f(-ct + \eta, t)$$

Integrate both sides with respect to t.

$$v(\eta, t) = \int_0^t f(-cs + \eta, s)\, ds + h(\eta),$$

where h is an arbitrary function of η. The lower limit of integration is arbitrary and has been set equal to 0. In order to change back to the original variable x, solve equation (9) for η.

$$x = -ct + \eta \quad \rightarrow \quad \eta = x + ct$$

Therefore,

$$v(x, t) = \int_0^t f(-cs + x + ct, s)\, ds + h(x + ct).$$

Part (d)

Substitute the solution for $v(x, t)$ into the one for $u(x, t)$.

$$\begin{aligned} u(x,t) &= \int_0^t v(cr + x - ct, r)\, dr + g(x - ct) \\ &= \int_0^t \left\{ \int_0^r f[-cs + (cr + x - ct) + cr, s]\, ds + h[(cr + x - ct) + cr] \right\} dr + g(x - ct) \\ &= \int_0^t \int_0^r f(x - ct + 2cr - cs, s)\, ds\, dr + \int_0^t h(x - ct + 2cr)\, dr + g(x - ct) \end{aligned}$$

Make the following substitution in the single integral.

$$\begin{aligned} p &= x - ct + 2cr \\ dp &= 2c\, dr \quad \rightarrow \quad \frac{1}{2c}\, dp = dr \end{aligned}$$

The formula for u becomes

$$\begin{aligned} u(x,t) &= \int_0^t \int_0^r f(x - ct + 2cr - cs, s)\, ds\, dr + \int_{x-ct}^{x+ct} h(p) \left(\frac{1}{2c}\, dp \right) + g(x - ct) \\ &= \int_0^t \int_0^r f(x - ct + 2cr - cs, s)\, ds\, dr + \frac{1}{2c} \int_{x-ct}^{x+ct} h(p)\, dp + g(x - ct) \\ &= \int_0^t \int_0^r f(x - ct + 2cr - cs, s)\, ds\, dr + \frac{1}{2c} H(x + ct) - \frac{1}{2c} H(x - ct) + g(x - ct). \end{aligned}$$

Use new arbitrary functions, $A(x-ct)$ and $B(x+ct)$, to simplify the expression.

$$u(x,t) = \int_0^t \int_0^r f(x-ct+2cr-cs,s)\,ds\,dr + A(x-ct) + B(x+ct)$$

In order to simplify the double integral we will switch the order of integration. At the moment, the inner integral is in ds, and s is present in both of f's arguments. r, on the other hand, is only in the first argument, so we can simplify the integrand if we make the inner integral in dr.

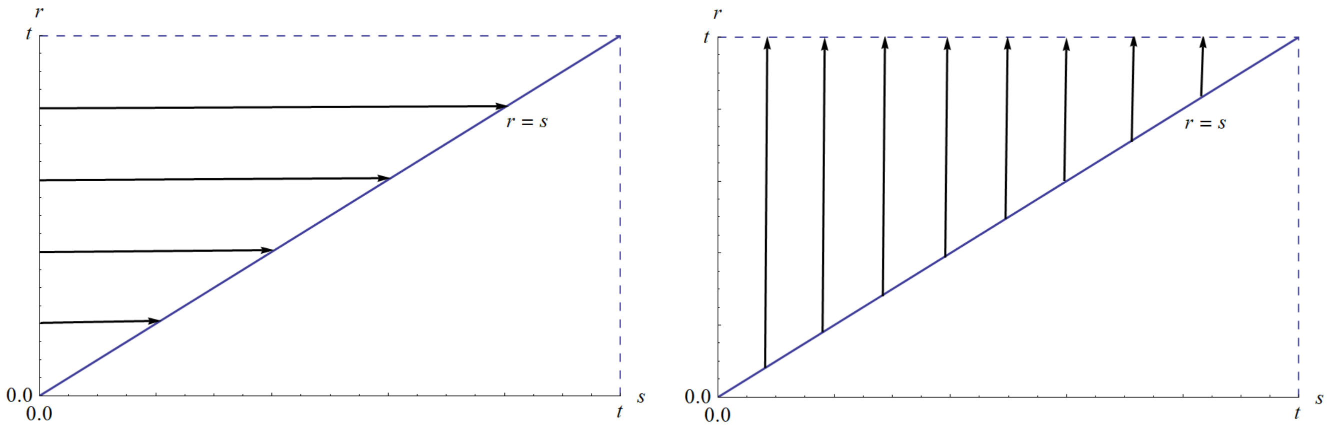

Figure 1: The current mode of integration in the sr-plane is shown on the left. This domain will be integrated over as shown on the right to simplify the integral.

$$u(x,t) = \int_0^t \int_s^t f(x-ct+2cr-cs,s)\,dr\,ds + A(x-ct) + B(x+ct)$$

Now the following substitution can be made in the integral.

$$y = x - ct + 2cr - cs$$
$$dy = 2c\,dr \quad \to \quad \frac{1}{2c}\,dy = dr$$

The result is

$$u(x,t) = \int_0^t \int_{x-ct+cs}^{x+ct-cs} f(y,s)\left(\frac{1}{2c}\,dy\right)\,ds + A(x-ct) + B(x+ct).$$

Therefore,

$$u(x,t) = A(x-ct) + B(x+ct) + \frac{1}{2c}\int_0^t \int_{x-c(t-s)}^{x+c(t-s)} f(y,s)\,dy\,ds.$$

This is the general solution to $u_{tt} = c^2 u_{xx} + f$. If we apply the two initial conditions, we can determine A and B.

Before doing so, take a derivative of the solution with respect to t.

$$\begin{aligned}
u_t(x,t) &= -cA'(x-ct) + cB'(x+ct) + \frac{1}{2c}\frac{\partial}{\partial t}\int_0^t \int_{x-c(t-s)}^{x+c(t-s)} f(y,s)\,dy\,ds \\
&= -cA'(x-ct) + cB'(x+ct) + \frac{1}{2c}\int_0^t \left[\frac{\partial}{\partial t}\int_{x-c(t-s)}^{x+c(t-s)} f(y,s)\,dy\right] ds + \underbrace{\int_x^x f(y,t)\,dy}_{=0} \\
&= -cA'(x-ct) + cB'(x+ct) + \frac{1}{2c}\int_0^t \left\{\int_{x-c(t-s)}^{x+c(t-s)} \underbrace{\frac{\partial}{\partial t} f(y,s)}_{=0}\,dy + f[x+c(t-s),s]\times(c) \right. \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \left. - f[x-c(t-s),s]\times(-c)\right\} ds \\
&= -cA'(x-ct) + cB'(x+ct) + \frac{1}{2}\int_0^t \{f[x+c(t-s),s] + f[x-c(t-s),s]\}\,ds
\end{aligned}$$

In differentiating the double integral, I made use of the Leibnitz integration rule which states that if

$$I(t) = \int_{a(t)}^{b(t)} \gamma(x,t)\,dx,$$

then

$$\frac{dI}{dt} = \int_{a(t)}^{b(t)} \frac{\partial \gamma}{\partial t}\,dx + \gamma[b(t),t]b'(t) - \gamma[a(t),t]a'(t).$$

From the initial conditions we obtain the following system of equations.

$$\begin{aligned}
u(x,0) &= A(x) + B(x) = \phi(x) \\
u_t(x,0) &= -cA'(x) + cB'(x) = \psi(x)
\end{aligned}$$

Even though this system is in terms of x, it's really in terms of w, where w is any expression we choose.

$$\begin{aligned}
A(w) + B(w) &= \phi(w) \\
-cA'(w) + cB'(w) &= \psi(w)
\end{aligned}$$

Differentiating both sides of the first equation with respect to w, we get

$$A'(w) + B'(w) = \phi'(w) \quad \to \quad B'(w) = \phi'(w) - A'(w).$$

Plug this expression for $B'(w)$ into the second equation.

$$-cA'(w) + c[\phi'(w) - A'(w)] = \psi(w) \quad \to \quad -2cA'(w) + c\phi'(w) = \psi(w) \quad \to \quad A'(w) = \frac{1}{2}\phi'(w) - \frac{1}{2c}\psi(w).$$

Solve for $A(w)$ and obtain an expression for $A(x-ct)$.

$$A(w) = \frac{1}{2}\phi(w) - \int^w \frac{1}{2c}\psi(s)\,ds + C_1 \quad \Rightarrow \quad A(x-ct) = \frac{1}{2}\phi(x-ct) - \int^{x-ct} \frac{1}{2c}\psi(s)\,ds + C_1$$

Use the first equation to solve for $B(w)$ and obtain an expression for $B(x+ct)$.

$$\begin{aligned}
B(w) &= \phi(w) - A(w) \\
&= \phi(w) - \frac{1}{2}\phi(w) + \int^w \frac{1}{2c}\psi(s)\,ds - C_1 \\
&= \frac{1}{2}\phi(w) + \int^w \frac{1}{2c}\psi(s)\,ds - C_1 \quad \Rightarrow \quad B(x+ct) = \frac{1}{2}\phi(x+ct) + \int^{x+ct} \frac{1}{2c}\psi(s)\,ds - C_1
\end{aligned}$$

The general solution for $u(x,t)$ becomes

$$\begin{aligned}
u(x,t) &= A(x-ct) + B(x+ct) + \frac{1}{2c}\int_0^t \int_{x-c(t-s)}^{x+c(t-s)} f(y,s)\,dy\,ds \\
&= \frac{1}{2}\phi(x-ct) - \int^{x-ct} \frac{1}{2c}\psi(s)\,ds + \cancel{C_1} + \frac{1}{2}\phi(x+ct) + \int^{x+ct} \frac{1}{2c}\psi(s)\,ds - \cancel{C_1} \\
&\qquad + \frac{1}{2c}\int_0^t \int_{x-c(t-s)}^{x+c(t-s)} f(y,s)\,dy\,ds \\
&= \frac{1}{2}[\phi(x+ct) + \phi(x-ct)] + \int_{x-ct} \frac{1}{2c}\psi(s)\,ds + \int^{x+ct} \frac{1}{2c}\psi(s)\,ds \\
&\qquad + \frac{1}{2c}\int_0^t \int_{x-c(t-s)}^{x+c(t-s)} f(y,s)\,dy\,ds \\
&= \frac{1}{2}[\phi(x+ct) + \phi(x-ct)] + \int_{x-ct}^{x+ct} \frac{1}{2c}\psi(s)\,ds + \frac{1}{2c}\int_0^t \int_{x-c(t-s)}^{x+c(t-s)} f(y,s)\,dy\,ds.
\end{aligned}$$

Therefore,

$$u(x,t) = \frac{1}{2}[\phi(x+ct) + \phi(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)\,ds + \frac{1}{2c}\int_0^t \int_{x-c(t-s)}^{x+c(t-s)} f(y,s)\,dy\,ds.$$