# Exercise 7

Let $A$ be a positive-definite $n \times n$ matrix. Let

$$S(t) = \sum_{m=0}^{\infty} \frac{(-1)^m A^{2m} t^{2m+1}}{(2m+1)!}.$$

(a) Show that this series of matrices converges uniformly for bounded $t$ and its sum $S(t)$ solves the problem $S''(t) + A^2 S(t) = 0$, $S(0) = 0$, $S'(0) = I$, where $I$ is the identity matrix. Therefore, it makes sense to denote $S(t)$ as $A^{-1} \sin tA$ and to denote its derivative $S'(t)$ as $\cos(tA)$.

(b) Show that the solution of (13) is (14).

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**Solution**

## Part (a)

Since the coefficients, 1 and $A^2$, are analytic for all $t$, the solution to the ODE $S''(t) + A^2 S(t) = 0$ can be represented as a power series centered at $t = 0$.

$$S(t) = \sum_{n=0}^{\infty} a_n t^n$$

We will substitute this expression into the ODE to determine $a_n$. Before doing so, take two derivatives to obtain a formula for $S''(t)$.

$$S(t) = \sum_{n=0}^{\infty} a_n t^n \quad \rightarrow \quad S'(t) = \sum_{n=0}^{\infty} n a_n t^{n-1} \quad \rightarrow \quad S''(t) = \sum_{n=0}^{\infty} n(n-1) a_n t^{n-2}$$

Plugging the series expansions for $S(t)$ and $S''(t)$ into the ODE, we get

$$\sum_{n=0}^{\infty} n(n-1) a_n t^{n-2} + A^2 \sum_{n=0}^{\infty} a_n t^n = 0.$$

Because of $n(n-1)$ in the first series, the index can be set to start from $n = 2$ without changing the value of the sum. Bring $A^2$ inside the second sum.

$$\sum_{n=2}^{\infty} n(n-1) a_n t^{n-2} + \sum_{n=0}^{\infty} A^2 a_n t^n = 0$$

In order to make the first series start from 0, make the substitution $k = n - 2$. Replace $n$ with $k$ in the second series to make the indices the same.

$$\sum_{k=0}^{\infty} (k+2)(k+1) a_{k+2} t^k + \sum_{k=0}^{\infty} A^2 a_k t^k = 0$$

Now that both series start from $k = 0$ and have $t^k$ in their summands, they can be combined.

$$\sum_{k=0}^{\infty} [(k+2)(k+1) a_{k+2} + A^2 a_k] t^k = 0$$

We thus have the following recurrence relation for the coefficients of the power series.

$$(k+2)(k+1)a_{k+2} + A^2 a_k = 0, \quad k = 0, 1, \ldots$$

Solve it for $a_{k+2}$.

$$a_{k+2} = -\frac{a_k}{(k+2)(k+1)}A^2, \quad k = 0, 1, \ldots$$

By testing this formula for various values of $k$, a general formula for $a_k$ can be deduced.

Even values of $k$

$$\begin{aligned}
k = 0: &\quad a_2 = -\frac{a_0}{2}A^2 \\
k = 2: &\quad a_4 = -\frac{a_2}{12}A^2 = \frac{a_0}{24}A^4 \\
k = 4: &\quad a_6 = -\frac{a_4}{30}A^2 = -\frac{a_0}{720}A^6 \\
k = 6: &\quad a_8 = -\frac{a_6}{56}A^2 = \frac{a_0}{40\,320}A^8 \\
\vdots &\quad \vdots \\
k = 2m-2: &\quad a_{2m} = (-1)^m \frac{a_0}{(2m)!}A^{2m}
\end{aligned}$$

Odd values of $k$

$$\begin{aligned}
k = 1: &\quad a_3 = -\frac{a_1}{6}A^2 \\
k = 3: &\quad a_5 = -\frac{a_3}{20}A^2 = \frac{a_1}{120}A^4 \\
k = 5: &\quad a_7 = -\frac{a_5}{42}A^2 = -\frac{a_1}{5\,040}A^6 \\
k = 7: &\quad a_9 = -\frac{a_7}{72}A^2 = \frac{a_1}{362\,880}A^8 \\
\vdots &\quad \vdots \\
k = 2m-1: &\quad a_{2m+1} = (-1)^m \frac{a_1}{(2m+1)!}A^{2m}
\end{aligned}$$

Consequently, we have for the general solution

$$\begin{aligned}
S(t) &= \sum_{n=0}^{\infty} a_n t^n = a_0 + a_1 t + a_2 t^2 + \cdots \\
&= \sum_{m=0}^{\infty} a_{2m} t^{2m} + \sum_{m=0}^{\infty} a_{2m+1} t^{2m+1} \\
&= \sum_{m=0}^{\infty} \frac{(-1)^m A^{2m} a_0}{(2m)!} t^{2m} + \sum_{m=0}^{\infty} \frac{(-1)^m A^{2m} a_1}{(2m+1)!} t^{2m+1} \\
&= a_0 \sum_{m=0}^{\infty} \frac{(-1)^m A^{2m}}{(2m)!} t^{2m} + a_1 \sum_{m=0}^{\infty} \frac{(-1)^m A^{2m}}{(2m+1)!} t^{2m+1}.
\end{aligned}$$

As expected there are two arbitrary constants, $a_0$ and $a_1$, due to the fact that the ODE is of the second order. The provided initial conditions, $S(0) = 0$ and $S'(0) = I$, can be used to determine them.

$$\begin{aligned}
S(0) &= a_0 = 0 \\
S'(0) &= a_1 = I
\end{aligned}$$

The general solution reduces to

$$\begin{aligned}
S(t) &= I \sum_{m=0}^{\infty} \frac{(-1)^m A^{2m}}{(2m+1)!} t^{2m+1} \\
&= \sum_{m=0}^{\infty} \frac{(-1)^m I A^{2m}}{(2m+1)!} t^{2m+1}.
\end{aligned}$$

$I$ is the identity matrix, so $IA^{2m} = A^{2m}$. Therefore,

$$\boxed{S(t) = \sum_{m=0}^{\infty} \frac{(-1)^m A^{2m}}{(2m+1)!} t^{2m+1}.}$$

Writing the series as

$$S(t) = \frac{1}{A} \sum_{m=0}^{\infty} \frac{(-1)^m A^{2m+1}}{(2m+1)!} t^{2m+1} = \frac{1}{A} \sum_{m=0}^{\infty} \frac{(-1)^m (At)^{2m+1}}{(2m+1)!},$$

we recognize that the infinite series is the Maclaurin series for sine.

$$S(t) = \frac{1}{A} \sin At$$

Differentiating both sides, we also have

$$S'(t) = \cos At.$$

To prove that the series representation for $S(t)$ converges uniformly, we will use the Weierstrass M-test, which states the following:[1]

*Suppose $\{f_n\}$ is a sequence of functions defined on $E$, and suppose*

$$|f_n(x)| \leq M_n \quad (x \in E,\ n = 1, 2, 3, \ldots).$$

*Then $\sum f_n$ converges uniformly on $E$ if $\sum M_n$ converges.*

The sequence of functions that we're interested in here is the summand of $S(t)$. Let

$$M_m = \left| \frac{(-1)^m A^{2m}}{(2m+1)!} t^{2m+1} \right| = \frac{A^{2m}}{(2m+1)!} t^{2m+1}.$$

The aim now is to show that

$$\sum_{m=0}^{\infty} M_m = \sum_{m=0}^{\infty} \frac{A^{2m} t^{2m+1}}{(2m+1)!}$$

is a convergent series. To do this, we will use the ratio test.

$$\begin{aligned} \lim_{m\to\infty} \left| \frac{M_{m+1}}{M_m} \right| = \lim_{m\to\infty} \frac{A^{2(m+1)} t^{2(m+1)+1}}{[2(m+1)+1]!} \cdot \frac{(2m+1)!}{A^{2m} t^{2m+1}} &= \lim_{m\to\infty} \frac{A^{2m+2} t^{2m+3}}{(2m+3)!} \cdot \frac{(2m+1)!}{A^{2m} t^{2m+1}} \\ &= \lim_{m\to\infty} \frac{A^2 t^2}{(2m+3)(2m+2)} = 0 \end{aligned}$$

Since the limit is less than 1, $\sum M_m$ is a convergent series. By the Weierstrass M-test, therefore, $S(t)$ converges uniformly. Furthermore, because the limit is equal to 0, $S(t)$ converges uniformly for all (bounded) $t$.

[1]This is from pg. 148 of the 3rd edition of Rudin's "Principles of Mathematical Analysis."

## Part (b)

Here we have to show that the solution to equation (13) in the text,

$$\frac{d^2u}{dt^2} + A^2u(t) = f(t), \quad u(0) = \phi, \quad \frac{du}{dt}(0) = \psi, \tag{13}$$

is given by equation (14),

$$u(t) = S'(t)\phi + S(t)\psi + \int_0^t S(t-s)f(s)\,ds, \tag{14}$$

where

$$S(t) = A^{-1}\sin tA \quad \text{and} \quad S'(t) = \cos tA.$$

## Solution by Reduction of Order

One way to solve this ODE is with reduction of order, where we determine the general solution from one of the solutions to the associated homogeneous equation,

$$\frac{d^2u}{dt^2} + A^2u(t) = 0.$$

Two solutions to this equation are $\cos At$ and $\sin At$, so we can choose to let

$$u(t) = v(t)\sin At.$$

Find $u''(t)$

$$\begin{aligned} u'(t) &= v'(t)\sin At + Av(t)\cos At \\ u''(t) &= v''(t)\sin At + Av'(t)\cos At + Av'(t)\cos At - A^2v(t)\sin At \\ &= v''(t)\sin At + 2Av'(t)\cos At - A^2v(t)\sin At \end{aligned}$$

and substitute these formulas into the original ODE.

$$[v''(t)\sin At + 2Av'(t)\cos At - \cancel{A^2v(t)\sin At}] + \cancel{A^2[v(t)\sin At]} = f(t)$$

$$(\sin At)v'' + 2A(\cos At)v' = f(t)$$

By making the substitution $w = v'$, the order of the ODE is thereby reduced from two to one.

$$(\sin At)w' + 2A(\cos At)w = f(t)$$

This ODE in particular can be solved by using an integrating factor. Put the equation in standard form by dividing both sides by $\sin At$.

$$w' + 2A\frac{\cos At}{\sin At}w = \frac{f(t)}{\sin At}$$

The integrating factor is then

$$I = \exp\left(\int^t 2A\frac{\cos As}{\sin As}\,ds\right) = \exp\left(\int^{\sin At} 2\frac{dy}{y}\right) = e^{2\ln|\sin At|} = \sin^2 At.$$

Multiply both sides of the ODE by $I$.

$$(\sin^2 At)w' + 2A(\sin At\cos At)w = f(t)\sin At$$

The left side can be written as $d/dt(Iw)$ now as a result of the product rule.

$$\frac{d}{dt}(w\sin^2 At) = f(t)\sin At$$

Integrate both sides with respect to $t$.

$$w\sin^2 At = \int_0^t f(s)\sin As\,ds + C_1,$$

where $C_1$ is an arbitrary constant. The lower limit of integration is arbitrary and has been set equal to 0. Divide both sides by $\sin^2 At$ to solve for $w$.

$$w(t) = \frac{1}{\sin^2 At}\int_0^t f(s)\sin As\,ds + \frac{C_1}{\sin^2 At}$$

Now replace $w(t)$ with $v'(t)$.

$$v'(t) = \frac{1}{\sin^2 At}\int_0^t f(s)\sin As\,ds + \frac{C_1}{\sin^2 At}$$

Integrate both sides with respect to $t$.

$$v(t) = \int^t \left[\frac{1}{\sin^2 Ar}\int_0^r f(s)\sin As\,ds + \frac{C_1}{\sin^2 Ar}\right] dr + C_2,$$

where $C_2$ is an arbitrary constant. Split the integral into two.

$$v(t) = \int^t \frac{1}{\sin^2 Ar}\int_0^r f(s)\sin As\,ds\,dr + \int^t \frac{C_1}{\sin^2 Ar}\,dr + C_2$$

Evaluate the single integral, bring $\sin^2 Ar$ inside the double integral, and set the lower limit of integration equal to 0.

$$v(t) = \int_0^t \int_0^r f(s)\csc^2 Ar\sin As\,ds\,dr - \frac{C_1}{A}\cot At + C_2$$

Since $r$ is only present in $\csc^2 Ar$, the double integral can be simplified if we make the inner integral in $dr$. Hence, the order of integration will be switched.

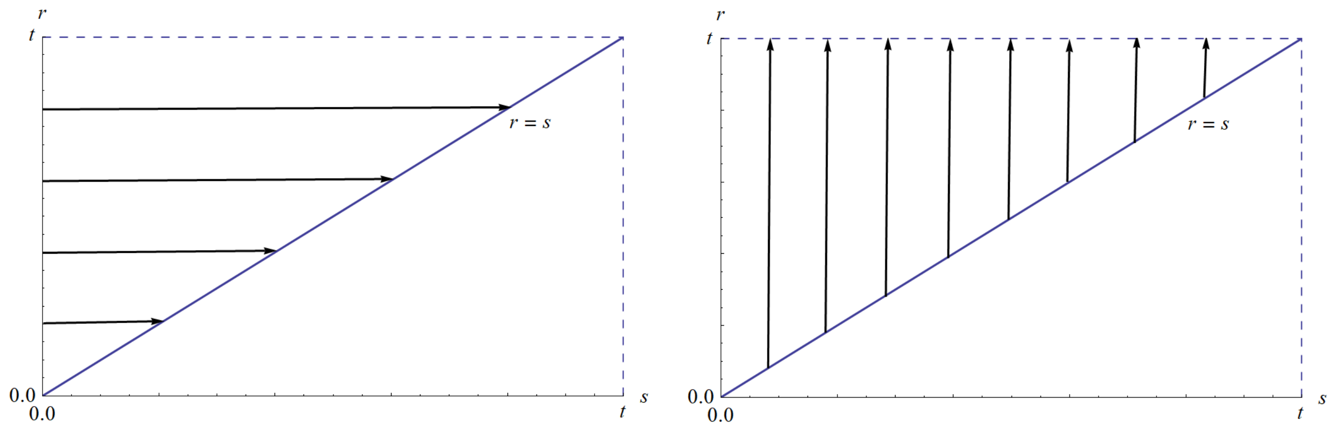


Figure 1: The current mode of integration in the $sr$-plane is shown on the left. This domain will be integrated over as shown on the right to simplify the integral.

$$v(t) = \int_0^t \int_s^t f(s)\csc^2 Ar \sin As\, dr\, ds - \frac{C_1}{A}\cot At + C_2$$
$$v(t) = \int_0^t f(s)\sin As \left(\int_s^t \csc^2 Ar\, dr\right) ds - \frac{C_1}{A}\cot At + C_2$$

Evaluate the integral in $dr$.

$$v(t) = \int_0^t f(s)\sin As \cdot \frac{1}{A}(-\cot Ar)\bigg|_s^t ds - \frac{C_1}{A}\cot At + C_2$$

Bring the constant in front of the integral.

$$v(t) = \frac{1}{A}\int_0^t f(s)\sin As(-\cot At + \cot As)\, ds - \frac{C_1}{A}\cot At + C_2$$

Write cotangent in terms of sine and cosine.

$$v(t) = \frac{1}{A}\int_0^t f(s)\sin As\left(-\frac{\cos At}{\sin At} + \frac{\cos As}{\sin As}\right) ds - \frac{C_1}{A}\frac{\cos At}{\sin At} + C_2$$

Combine the fractions in the parentheses.

$$v(t) = \frac{1}{A}\int_0^t f(s)\sin As\left(\frac{-\cos At\sin As + \sin At\cos As}{\sin At\sin As}\right) ds - \frac{C_1}{A}\frac{\cos At}{\sin At} + C_2$$

We then have for $u(t)$

$$\begin{aligned} u(t) &= v(t)\sin At \\ &= \sin At\left[\frac{1}{A}\int_0^t f(s)\sin As\left(\frac{-\cos At\sin As + \sin At\cos As}{\sin At\sin As}\right) ds - \frac{C_1}{A}\frac{\cos At}{\sin At} + C_2\right] \\ &= \frac{1}{A}\int_0^t f(s)\cancel{\sin At\sin As}\left(\frac{-\cos At\sin As + \sin At\cos As}{\cancel{\sin At\sin As}}\right) ds - \frac{C_1}{A}\cancel{\sin At}\frac{\cos At}{\cancel{\sin At}} + C_2\sin At. \end{aligned}$$

What remains after the cancellation is

$$u(t) = \frac{1}{A}\int_0^t f(s)(\sin At\cos As - \cos At\sin As)\,ds - \frac{C_1}{A}\cos At + C_2\sin At.$$

From the angle subtraction formula for sine, the integrand can be written as

$$u(t) = \frac{1}{A}\int_0^t f(s)\sin(At - As)\,ds - \frac{C_1}{A}\cos At + C_2\sin At.$$

The constants of integration, $C_1$ and $C_2$, can be determined from the provided initial conditions. Before using them, though, take a derivative of $u(t)$ first to find $u'(t)$.

$$\begin{aligned} u'(t) &= \frac{1}{A}\left[\int_0^t f(s)\frac{\partial}{\partial t}\sin(At - As)\,ds + f(t)\sin 0 \times 1 - f(0)\sin At \times 0\right] + C_1\sin At + C_2 A\cos At \\ &= \int_0^t f(s)\cos(At - As)\,ds + C_1\sin At + C_2 A\cos At \end{aligned}$$

In differentiating the integral, I made use of the Leibnitz integration rule which states that if

$$I(t) = \int_{a(t)}^{b(t)} \gamma(x,t)\,dx,$$

then

$$\frac{dI}{dt} = \int_{a(t)}^{b(t)} \frac{\partial \gamma}{\partial t}\,dx + \gamma[b(t),t]b'(t) - \gamma[a(t),t]a'(t).$$

From the initial conditions we obtain the following system of equations.

$$\begin{aligned} u(0) &= -\frac{C_1}{A} = \phi \\ u'(0) &= C_2 A = \psi \qquad \rightarrow \qquad C_2 = \frac{\psi}{A} \end{aligned}$$

The solution for $u(t)$ becomes

$$u(t) = \frac{1}{A}\int_0^t f(s)\sin(At - As)\,ds + \phi\cos At + \frac{\psi}{A}\sin At.$$

Therefore,

$$u(t) = S'(t)\phi + S(t)\psi + \int_0^t S(t-s)f(s)\,ds,$$

where

$$S(t) = A^{-1}\sin tA \quad \text{and} \quad S'(t) = \cos tA.$$

## Solution by Variation of Parameters

Another way to solve equation (13) is with the method of variation of parameters.

$$\frac{d^2u}{dt^2} + A^2u(t) = f(t), \quad u(0) = \phi, \quad \frac{du}{dt}(0) = \psi$$

Because this is a second-order linear inhomogeneous ODE, the general solution can be written as the sum of a complementary solution $u_c$ and a particular solution $u_p$,

$$u = u_c + u_p.$$

$u_c$ is the solution to the associated homogeneous ODE.

$$\frac{d^2u_c}{dt^2} + A^2u_c(t) = 0$$

It can be written in terms of sine and cosine.

$$u_c(t) = C_3 \cos At + C_4 \sin At$$

According to the variation of parameters method, the particular solution is obtained by allowing the constants to vary as functions of $t$.

$$u_p(t) = u_1(t) \cos At + u_2(t) \sin At$$

This expression will be substituted into the original ODE to determine $u_1$ and $u_2$. To do that, find $u_p''(t)$.

$$\begin{aligned} u_p'(t) &= u_1'(t) \cos At - Au_1(t) \sin At + u_2'(t) \sin At + Au_2(t) \cos At \\ &= \cos At[u_1'(t) + Au_2(t)] + \sin At[-Au_1(t) + u_2'(t)] \\ &= \cos At(u_1' + Au_2) + \sin At(-Au_1 + u_2') \end{aligned}$$

$$\begin{aligned} u_p''(t) &= -A \sin At(u_1' + Au_2) + \cos At(u_1'' + Au_2') + A \cos At(-Au_1 + u_2') + \sin At(-Au_1' + u_2'') \\ &= \sin At(-Au_1' - A^2u_2 - Au_1' + u_2'') + \cos At(u_1'' + Au_2' - A^2u_1 + Au_2') \\ &= \sin At(u_2'' - 2Au_1' - A^2u_2) + \cos At(u_1'' + 2Au_2' - A^2u_1) \end{aligned}$$

So we have

$$[\sin At(u_2'' - 2Au_1' - \cancel{A^2u_2}) + \cos At(u_1'' + 2Au_2' - \cancel{A^2u_1})] + A^2(\cancel{u_1 \cos At} + \cancel{u_2 \sin At}) = f(t).$$

Simplify the left side.

$$u_2'' \sin At + u_1'' \cos At + 2A(-u_1' \sin At + u_2' \cos At) = f(t)$$

Since there is only one ODE and two unknowns, $u_1$ and $u_2$, we are free to create an additional equation relating them. By letting

$$-u_1' \sin At + u_2' \cos At = 0, \tag{1}$$

the ODE reduces to

$$u_2'' \sin At + u_1'' \cos At = f(t). \tag{2}$$

Solve equation (1) for $u_2'$

$$u_2' = \frac{\sin At}{\cos At} u_1' = u_1' \tan At \tag{3}$$

and substitute the expression into equation (2) to get an ODE for $u_1$.

$$(u_1' \tan At)' \sin At + u_1'' \cos At = f(t)$$

Evaluate the derivative on the left side with the product rule.

$$(u_1'' \tan At + Au_1' \sec^2 At) \sin At + u_1'' \cos At = f(t)$$

Factor the left side.

$$u_1''(\sin At \tan At + \cos At) + u_1'(A \sec^2 At \sin At) = f(t)$$

Factor $\sec At$ from the first term and change $\sec At \sin At$ to $\tan At$ in the second term.

$$(\sec At)u_1''(\sin^2 At + \cos^2 At) + (A \sec At \tan At)u_1' = f(t)$$

Use the fact that $\sin^2 At + \cos^2 At = 1$.

$$(\sec At)u_1'' + (A \sec At \tan At)u_1' = f(t)$$

As a result of the product rule, the left side can be written as $d/dt(u_1' \sec At)$.

$$\frac{d}{dt}(u_1' \sec At) = f(t)$$

Integrate both sides with respect to $t$.

$$u_1' \sec At = \int_0^t f(s)\, ds,$$

The constant of integration and the lower limit of integration are arbitrary and have been set equal to 0. Multiply both sides by $\cos At$.

$$u_1' = \cos At \int_0^t f(s)\, ds \tag{4}$$

Integrate both sides with respect to $t$ once more to solve for $u_1$.

$$\begin{aligned} u_1(t) &= \int^t \cos Ar \int_0^r f(s)\, ds\, dr \\ &= \int_0^t \int_0^r f(s) \cos Ar\, ds\, dr \end{aligned}$$

Again, the constant of integration and the lower limit of integration are arbitrary and have been set equal to 0. We see that $\cos Ar$ can be integrated, so we switch the order or integration to make $dr$ come first. (See Figure 1.)

$$\begin{aligned} u_1(t) &= \int_0^t \int_s^t f(s) \cos Ar\, dr\, ds \\ &= \int_0^t f(s) \left( \int_s^t \cos Ar\, dr \right) ds \\ &= \int_0^t f(s) \cdot \frac{1}{A} (\sin Ar) \Big|_s^t\, ds \\ &= \frac{1}{A} \int_0^t f(s)(\sin At - \sin As)\, ds \end{aligned}$$

Substitute the formula for $u_1'$ in equation (4) into equation (3) to obtain an ODE for $u_2'$.

$$\begin{aligned} u_2'(t) &= u_1' \tan At \\ &= \sin At \int_0^t f(s)\,ds \end{aligned}$$

Integrate both sides with respect to $t$.

$$\begin{aligned} u_2(t) &= \int^t \sin Ar \int_0^r f(s)\,ds\,dr \\ &= \int_0^t \int_0^r f(s) \sin Ar\,ds\,dr \end{aligned}$$

The constant of integration and the lower limit of integration are arbitrary and have been set equal to 0. We see that $\sin Ar$ can be integrated, so we switch the order or integration to make $dr$ come first. (See Figure 1.)

$$\begin{aligned} u_2(t) &= \int_0^t \int_s^t f(s) \sin Ar\,dr\,ds \\ &= \int_0^t f(s) \left( \int_s^t \sin Ar\,dr \right) ds \\ &= \int_0^t f(s) \cdot \frac{1}{A}(-\cos Ar)\bigg|_s^t ds \\ &= -\frac{1}{A} \int_0^t f(s)(\cos At - \cos As)\,ds \end{aligned}$$

The particular solution is thus

$$\begin{aligned} u_p(t) &= u_1(t) \cos At + u_2(t) \sin At \\ &= \frac{1}{A} \cos At \int_0^t f(s)(\sin At - \sin As)\,ds - \frac{1}{A} \sin At \int_0^t f(s)(\cos At - \cos As)\,ds \\ &= \frac{1}{A} \int_0^t f(s)(\cancel{\sin At \cos At} - \sin As \cos At - \cancel{\sin At \cos At} + \sin At \cos As)\,ds \\ &= \frac{1}{A} \int_0^t f(s)(\sin At \cos As - \sin As \cos At)\,ds \\ &= \frac{1}{A} \int_0^t f(s) \sin(At - As)\,ds \end{aligned}$$

So then we have for the general solution

$$u(t) = C_3 \cos At + C_4 \sin At + \frac{1}{A} \int_0^t f(s) \sin(At - As)\,ds.$$

Applying the initial conditions here determines $C_3$ and $C_4$.

$$\begin{aligned} u(0) &= C_3 = \phi \\ u'(0) &= C_4 A = \psi \qquad \rightarrow \qquad C_4 = \frac{\psi}{A} \end{aligned}$$

Therefore, the solution of (13) is (14).