Exercise 10

Use any method to show that \( u = 1/(2c) \int_D f \) solves the inhomogeneous wave equation on the half-line with zero initial and boundary data, where \( D \) is the domain of dependence for the half-line.

Solution

The initial boundary value problem to solve is as follows.

\[
\begin{align*}
  u_{tt} - c^2 u_{xx} &= f(x,t), \quad 0 < x < \infty, \ t > 0 \\
  u(x,0) &= 0 \quad u_t(x,0) = 0 \\
  u(0,t) &= 0
\end{align*}
\]

Comparing the wave equation to the general form of a second-order PDE,

\[ Au_{tt} + Bu_{xt} + Cu_{xx} + Du_t + Eu_x + Fu = G, \]

we see that \( A = 1, \ B = 0, \ C = -c^2, \ D = 0, \ E = 0, \ F = 0, \) and \( G = f(x,t) \). The characteristic equations for the second-order PDE are given by

\[
\begin{align*}
  \frac{dx}{dt} &= \frac{1}{2A} (B \pm \sqrt{B^2 - 4AC}) \\
  &= \frac{1}{2}(\pm \sqrt{4c^2}) \\
  &= \pm c.
\end{align*}
\]

Because the discriminant \( B^2 - 4AC = 4c^2 \) is positive, the two families of characteristic curves are real and distinct. In particular, they are lines with slopes \( \pm c \) and characteristic coordinates, \( \xi \) and \( \eta \), respectively.

\[
\begin{align*}
  \frac{dx}{dt} &= c \quad \rightarrow \quad x = ct + \xi \\
  \frac{dx}{dt} &= -c \quad \rightarrow \quad x = -ct + \eta
\end{align*}
\]

Suppose we are interested in evaluating \( u \) at the point \( (x_0,t_0) \). The equations of the lines going through this point are

\[
\begin{align*}
  x - x_0 &= c(t - t_0) \\
  x - x_0 &= -c(t - t_0)
\end{align*}
\]

As shown in the figure below, if \( (x_0,t_0) \) lies in the domain \( x + ct > 0 \), then the solution behaves as if there were no boundary. On the other hand, if \( (x_0,t_0) \) lies in the domain \( x - ct < 0 \), then a reflection occurs at the boundary. The solution has to be considered in each case.
Figure 1: The presence of a boundary at \( x = 0 \) means we have to consider the solution to the PDE in the domains above and below the line \( x - ct = 0 \). The reason is that a reflection occurs for points above it but not below it.

**Case 1: \( x - ct > 0 \)**

No reflection occurs in this case. Integrate both sides of the PDE over the triangular domain \( D_1 \) enclosed by the lines (from left to right as indicated below).

Write the double integral explicitly on the right side.

\[
\iint_{D_1} (u_{tt} - c^2 u_{xx}) \, dA = \int_0^t \int_{x_0+c(t-t_0)}^{x_0-c(t-t_0)} f(x,t) \, dx \, dt
\]

The integral will be implicit from here until the end to save space. Rewrite the left side.

\[
-\iint_{D_1} \left[ \frac{\partial}{\partial x} (c^2 u_x) - \frac{\partial}{\partial t} (u_t) \right] \, dA = \iint_{D_1} f(x,t) \, dx \, dt
\]
Multiply both sides by $-1$.

$$\iint_{D_1} \left[ \frac{\partial}{\partial x} (c^2 u_x) - \frac{\partial}{\partial t} (u_t) \right] dA = - \iint_{D_1} f(x,t) \, dx \, dt$$

Apply Green’s theorem (essentially the divergence theorem in two dimensions) to the double integral on the left to turn it into a counterclockwise line integral around the triangle’s boundary $\text{bdy } D_1$.

$$\oint_{\text{bdy } D_1} (u_t \, dx + c^2 u_x \, dt) = - \iint_{D_1} f(x,t) \, dx \, dt$$

Let $L_1$, $L_2$, and $L_3$ represent the legs of the triangle.

The line integral is the sum of three integrals, one over each leg.

$$\int_{L_1} (u_t \, dx + c^2 u_x \, dt) + \int_{L_2} (u_t \, dx + c^2 u_x \, dt) + \int_{L_3} (u_t \, dx + c^2 u_x \, dt) = - \iint_{D_1} f(x,t) \, dx \, dt$$

On $L_1$

- $t = 0$
- $dt = 0$
- $dx = -c \, dt$

On $L_2$

- $x - x_0 = -c(t - t_0)$
- $dx = c \, dt$

On $L_3$

- $x - x_0 = c(t - t_0)$
- $dx = c \, dt$

Replace the differentials in the integrals over $L_2$ and $L_3$.

$$\int_{x_0-ct_0}^{x_0+ct_0} u_t(x,0) \, dx + \int_{L_2} \left[ u_t(-c \, dt) + c^2 u_x \left( -\frac{dx}{c} \right) \right] + \int_{L_3} \left[ u_t(c \, dt) + c^2 u_x \left( \frac{dx}{c} \right) \right]$$

$$= - \iint_{D_1} f(x,t) \, dx \, dt$$

In this exercise $u_t(x,0) = 0$, so the integral over $L_1$ vanishes.

$$-c \int_{L_2} \left( \frac{\partial u}{\partial t} \, dt + \frac{\partial u}{\partial x} \, dx \right) + c \int_{L_3} \left( \frac{\partial u}{\partial t} \, dt + \frac{\partial u}{\partial x} \, dx \right) = - \iint_{D_1} f(x,t) \, dx \, dt$$

The integrands on the left side are how the differential of $u = u(x,t)$ is defined.

$$-c \int_{L_2} du + c \int_{L_3} du = - \iint_{D_1} f(x,t) \, dx \, dt$$

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Evaluate the integrals on the left side.

\[ -c[u(x_0, t_0) - u(x_0 + ct_0, 0)] + c[u(x_0 - ct_0, 0) - u(x_0, t_0)] = -\iint_{D_1} f(x, t) \, dx \, dt \]

In this exercise \( u(x, 0) = 0 \), so \( u(x_0 + ct_0, 0) = 0 \) and \( u(x_0 - ct_0, 0) = 0 \).

\[ -2cu(x_0, t_0) = -\iint_{D_1} f(x, t) \, dx \, dt \]

Divide both sides by \(-2c\) and write the double integral explicitly again.

\[ u(x_0, t_0) = \frac{1}{2c} \int_0^{t_0} \int_{x_0 + c(t-t_0)}^{x_0 - c(t-t_0)} f(x, t) \, dx \, dt \]

Finally, switch the roles of \( x \) and \( t \) with those of \( x_0 \) and \( t_0 \), respectively.

\[ u(x, t) = \frac{1}{2c} \int_0^t \int_{x-c(t-t_0)}^{x+c(t-t_0)} f(x, t) \, dx_0 \, dt_0 \]

Therefore,

\[ u(x, t) = \frac{1}{2c} \int_0^t \int_{x-c(t-t_0)}^{x+c(t-t_0)} f(x, t) \, dx_0 \, dt_0, \quad x - ct > 0. \]

**Case 2: \( x - ct < 0 \)**

Integrate both sides of the PDE over the polygonal domain \( D_2 \) enclosed by the lines (from left to right as indicated below).

Write the double integral explicitly on the right side.

\[ \iint_{D_2} (u_{tt} - c^2 u_{xx}) \, dA = \int_{t_0}^{0} \int_{x_0+c(t-t_0)}^{x_0-c(t-t_0)} f(x, t) \, dx \, dt + \int_{0}^{t_0} \int_{x_0-c(t-t_0)}^{x_0+c(t-t_0)} f(x, t) \, dx \, dt \]

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The integral will be implicit from here until the end to save space. Rewrite the left side.

\[- \iint_{D_2} \left[ \frac{\partial}{\partial x} (c^2 u_x) - \frac{\partial}{\partial t} (u_t) \right] \, dA = \iint_{D_2} f(x, t) \, dx \, dt\]

Multiply both sides by \(-1\).

\[\iint_{D_2} \left[ \frac{\partial}{\partial x} (c^2 u_x) - \frac{\partial}{\partial t} (u_t) \right] \, dA = - \iint_{D_2} f(x, t) \, dx \, dt\]

Apply Green’s theorem to the double integral on the left to turn it into a counterclockwise line integral around the polygon’s boundary \(bDY_{D_2}\).

\[\oint_{bDY_{D_2}} (u_t \, dx + c^2 u_x \, dt) = - \iint_{D_2} f(x, t) \, dx \, dt\]

Let \(L_4, L_5, L_6, \) and \(L_7\) represent the legs of the polygon.

The line integral is the sum of four integrals, one over each leg.

\[\int_{L_4} (u_t \, dx + c^2 u_x \, dt) + \int_{L_5} (u_t \, dx + c^2 u_x \, dt) + \int_{L_6} (u_t \, dx + c^2 u_x \, dt) + \int_{L_7} (u_t \, dx + c^2 u_x \, dt) = - \iint_{D_2} f(x, t) \, dx \, dt\]

On \(L_4\):
- \(t = 0\)
- \(x = x_0 - c(t - t_0)\)
- \(dt = 0\)
- \(dx = -c \, dt\)

On \(L_5\):
- \(x = x_0\)
- \(x = x_0 + c(t - t_0)\)
- \(dx = c \, dt\)

On \(L_6\):
- \(x = -x_0 - c(t - t_0)\)
- \(dx = -c \, dt\)

On \(L_7\):
- \(x = x_0 - c(t - t_0)\)
- \(dx = -c \, dt\)

Replace the differentials in the integrals over \(L_5, L_6, \) and \(L_7\).

\[\int_{x_0-c(t_0)}^{x_0+c(t_0)} u_t(x, 0) \, dx + \int_{L_5} \left[ u_t(-c \, dt) + c^2 u_x \left( -\frac{dx}{c} \right) \right] + \int_{L_6} \left[ u_t(c \, dt) + c^2 u_x \left( \frac{dx}{c} \right) \right] + \int_{L_7} \left[ u_t(-c \, dt) + c^2 u_x \left( -\frac{dx}{c} \right) \right] = - \iint_{D_2} f(x, t) \, dx \, dt\]

In this exercise \(u_t(x, 0) = 0\), so the integral over \(L_4\) vanishes.

\[-c \int_{L_5} \left( \frac{\partial u}{\partial t} \, dt + \frac{\partial u}{\partial x} \, dx \right) + c \int_{L_6} \left( \frac{\partial u}{\partial t} \, dt + \frac{\partial u}{\partial x} \, dx \right) - c \int_{L_7} \left( \frac{\partial u}{\partial t} \, dt + \frac{\partial u}{\partial x} \, dx \right) = - \iint_{D_2} f(x, t) \, dx \, dt\]
In this exercise $u = u(x, t)$ is defined.

$$-c \int_{L_5} du + c \int_{L_6} du - c \int_{L_7} du = - \int_{D_2} f(x, t) \, dx \, dt$$

Evaluate the integrals on the left side.

$$-c[u(x_0, t_0) - u(x_0 + ct_0, 0)] + c \left[ u \left( 0, t_0 - \frac{x_0}{c} \right) - u(x_0, t_0) \right] - c \left[ u(ct_0 - x_0, 0) - u \left( 0, t_0 - \frac{x_0}{c} \right) \right]$$

$$= - \int_{D_2} f(x, t) \, dx \, dt$$

In this exercise $u(x, 0) = 0$ and $u(0, t) = 0$, so $u(x_0 + ct_0, 0) = 0$ and $u(ct_0 - x_0, 0) = 0$ and $u(0, t_0 - x_0/c) = 0$.

$$-2cu(x_0, t_0) = - \int_{D_2} f(x, t) \, dx \, dt$$

Divide both sides by $-2c$ and write out the double integral explicitly again.

$$u(x_0, t_0) = \frac{1}{2c} \left[ \int_{t_0}^{t_0 - \frac{x_0}{c}} \int_{x_0}^{x_0 - c(t-t_0)} f(x, t) \, dx \, dt + \int_{t_0}^{t_0 - \frac{x_0}{c}} \int_{-c(t-t_0)}^{c(t-t_0)} f(x, t) \, dx \, dt \right]$$

Finally, switch the roles of $x$ and $t$ with those of $x_0$ and $t_0$, respectively.

$$u(x, t) = \frac{1}{2c} \left[ \int_{t - \frac{x}{c}}^{t} \int_{x}^{x - c(t-t_0)} f(x_0, t_0) \, dx_0 \, dt_0 + \int_{t - \frac{x}{c}}^{t} \int_{-c(t-t_0)}^{c(t-t_0)} f(x_0, t_0) \, dx_0 \, dt_0 \right]$$

Therefore,

$$u(x, t) = \frac{1}{2c} \left[ \int_{t - \frac{x}{c}}^{t} \int_{x - c(t-t_0)}^{x + c(t-t_0)} f(x_0, t_0) \, dx_0 \, dt_0 + \int_{t - \frac{x}{c}}^{t} \int_{-c(t-t_0)}^{c(t-t_0)} f(x_0, t_0) \, dx_0 \, dt_0 \right], \quad x - ct < 0.$$