Exercise 13

Solve \( u_{tt} = c^2 u_{xx} \) for \( 0 < x < \infty \),
\[ u(0, t) = t^2, \quad u(x, 0) = x, \quad u_t(x, 0) = 0. \]

Solution

It will be assumed that \( t > 0 \). Bring \( c^2 u_{xx} \) to the left side.

\[ u_{tt} - c^2 u_{xx} = 0 \]

Comparing the wave equation to the general form of a second-order PDE,

\[ Au_{tt} + Bu_{xt} + Cu_{xx} + Du_t + Eu_x + Fu = G, \]

we see that \( A = 1, B = 0, C = -c^2, D = 0, E = 0, F = 0, \) and \( G = 0 \). The characteristic equations for a second-order PDE are given by

\[ \frac{dx}{dt} = \frac{1}{2A} (B \pm \sqrt{B^2 - 4AC}) \]
\[ = \frac{1}{2} (\pm \sqrt{4c^2}) \]
\[ = \pm c. \]

Because the discriminant \( B^2 - 4AC = 4c^2 \) is positive, the two families of characteristic curves are real and distinct. In particular, they are lines with slopes \( \pm c \) and characteristic coordinates, \( \xi \) and \( \eta \), respectively.

\[ \frac{dx}{dt} = c \quad \rightarrow \quad x = ct + \xi \]
\[ \frac{dx}{dt} = -c \quad \rightarrow \quad x = -ct + \eta \]

Suppose we are interested in evaluating \( u \) at the point \((x_0, t_0)\). The equations of the lines going through this point are

\[ x - x_0 = c(t - t_0) \]
\[ x - x_0 = -c(t - t_0). \]

As shown in the figure below, if \((x_0, t_0)\) lies in the domain \( x + ct > 0 \), then the solution behaves as if there were no boundary. On the other hand, if \((x_0, t_0)\) lies in the domain \( x - ct < 0 \), then a reflection occurs at the boundary. The solution has to be considered in each case.
Figure 1: The presence of a boundary at $x = 0$ means we have to consider the solution to the PDE in the domains above and below the line $x - ct = 0$. The reason is that a reflection occurs for points above it but not below it.

**Case 1: $x - ct > 0$**

No reflection occurs in this case. Integrate both sides of the PDE over the triangular domain $D_1$ enclosed by the lines (from left to right as indicated above).

\[ \iint_{D_1} (u_{tt} - c^2 u_{xx}) \, dA = 0 \]

Rewrite the left side.

\[ -\iint_{D_1} \left[ \frac{\partial}{\partial x} (c^2 u_x) - \frac{\partial}{\partial t} (u_t) \right] \, dA = 0 \]

Multiply both sides by $-1$. 

\[ \iint_{D_1} \left[ \frac{\partial}{\partial x} (c^2 u_x) - \frac{\partial}{\partial t} (u_t) \right] \, dA = 0 \]

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Apply Green’s theorem (essentially the divergence theorem in two dimensions) to the double integral to turn it into a counterclockwise line integral around the triangle’s boundary bdy $D_1$.

$$\oint_{bdy\,D_1} (u_t \, dx + c^2u_x \, dt) = 0$$

Let $L_1$, $L_2$, and $L_3$ represent the legs of the triangle.

The line integral is the sum of three integrals, one over each leg.

$$\int_{L_1} (u_t \, dx + c^2u_x \, dt) + \int_{L_2} (u_t \, dx + c^2u_x \, dt) + \int_{L_3} (u_t \, dx + c^2u_x \, dt) = 0$$

On $L_1$  
$t = 0$  
$dt = 0$

On $L_2$  
$x - x_0 = -c(t - t_0)$  
$dx = -c \, dt$

On $L_3$  
$x - x_0 = c(t - t_0)$  
$dx = c \, dt$

Replace the differentials in the integrals over $L_2$ and $L_3$.

$$\int_{x_0 - ct_0}^{x_0 + ct_0} u_t(x, 0) \, dx + \int_{L_2} [u_t(-c \, dt) + c^2u_x \left( -\frac{dx}{c} \right)] + \int_{L_3} [u_t(c \, dt) + c^2u_x \left( \frac{dx}{c} \right)] = 0$$

In this exercise $u_t(x, 0) = 0$, so the integral over $L_1$ vanishes.

$$-c \int_{L_2} \left( \frac{\partial u}{\partial t} \, dt + \frac{\partial u}{\partial x} \, dx \right) + c \int_{L_3} \left( \frac{\partial u}{\partial t} \, dt + \frac{\partial u}{\partial x} \, dx \right) = 0$$

The remaining integrands are how the differential of $u = u(x, t)$ is defined.

$$-c \int_{L_2} du + c \int_{L_3} du = 0$$

Evaluate the remaining integrals.

$$-c[u(x_0, t_0) - u(x_0 + ct_0, 0)] + c[u(x_0 - ct_0, 0) - u(x_0, t_0)] = 0$$

In this exercise $u(x, 0) = x$, so $u(x_0 + ct_0, 0) = x_0 + ct_0$ and $u(x_0 - ct_0, 0) = x_0 - ct_0$.

$$-2cu(x_0, t_0) + c[(x_0 + ct_0) + (x_0 - ct_0)] = 0$$

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Solve this equation for $2cu(x_0, t_0)$.

$$2cu(x_0, t_0) = c[(x_0 + ct_0) + (x_0 - ct_0)]$$

$$= 2cx_0$$

Divide both sides by $2c$.

$$u(x_0, t_0) = x_0$$

Therefore, switching the roles of $x$ and $t$ with those of $x_0$ and $t_0$, respectively,

$$u(x, t) = x, \quad x - ct > 0.$$ 

**Case 2: $x - ct < 0$**

Integrate both sides of the PDE over the polygonal domain $D_2$ enclosed by the lines (from left to right as indicated above).

$$\iint_{D_2} (u_{tt} - c^2 u_{xx}) \, dA = 0$$

Rewrite the left side.

$$-\iint_{D_2} \left[ \frac{\partial}{\partial x} (c^2 u_x) - \frac{\partial}{\partial t} (u_t) \right] \, dA = 0$$

Multiply both sides by $-1$.

$$\iint_{D_2} \left[ \frac{\partial}{\partial x} (c^2 u_x) - \frac{\partial}{\partial t} (u_t) \right] \, dA = 0$$

Apply Green’s theorem to the double integral to turn it into a counterclockwise line integral around the polygon’s boundary $\text{bdy } D_2$.

$$\oint_{\text{bdy } D_2} (u_t \, dx + c^2 u_x \, dt) = 0$$

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Let $L_4$, $L_5$, $L_6$, and $L_7$ represent the legs of the polygon.

The line integral is the sum of four integrals, one over each leg.

\[
\int_{L_4} (u_t \, dx + c^2 u_x \, dt) + \int_{L_5} (u_t \, dx + c^2 u_x \, dt) + \int_{L_6} (u_t \, dx + c^2 u_x \, dt) + \int_{L_7} (u_t \, dx + c^2 u_x \, dt) = 0
\]

On $L_4$
\[
t = 0 \quad x - x_0 = -c(t - t_0)
\]
\[
dt = 0 \quad dx = -c \, dt
\]

On $L_5$
\[
t = 0 \quad x - x_0 = c(t - t_0)
\]
\[
dx = c \, dt
\]

On $L_6$
\[
t = 0 \quad x = -x_0 - c(t - t_0)
\]
\[
dx = -c \, dt
\]

On $L_7$
\[
t = 0 \quad x = -x_0 - c(t - t_0)
\]
\[
dx = -c \, dt
\]

Replace the differentials in the integrals over $L_5$, $L_6$, and $L_7$.

\[
\int_{x_0 + ct_0}^{x_0} u_t(x, 0) \, dx + \int_{L_5} \left[ u_t(-c \, dt) + c^2 u_x \left( \frac{dx}{c} \right) \right] + \int_{L_6} \left[ u_t(c \, dt) + c^2 u_x \left( \frac{dx}{c} \right) \right] + \int_{L_7} \left[ u_t(-c \, dt) + c^2 u_x \left( \frac{dx}{c} \right) \right] = 0
\]

In this exercise $u_t(x, 0) = 0$, so the integral over $L_4$ vanishes.

\[
-c \int_{L_5} \left( \frac{\partial u}{\partial t} \, dt + \frac{\partial u}{\partial x} \, dx \right) + c \int_{L_6} \left( \frac{\partial u}{\partial t} \, dt + \frac{\partial u}{\partial x} \, dx \right) - c \int_{L_7} \left( \frac{\partial u}{\partial t} \, dt + \frac{\partial u}{\partial x} \, dx \right) = 0
\]

The remaining integrands on the left side are how the differential of $u = u(x,t)$ is defined.

\[
-c \int_{L_5} \, du + c \int_{L_6} \, du - c \int_{L_7} \, du = 0
\]

Evaluate the remaining integrals.

\[
-c[u(x_0, t_0) - u(x_0 + ct_0, 0)] + c \left[ u \left( 0, t_0 - \frac{x_0}{c} \right) - u(x_0, t_0) \right] - c \left[ u(ct_0 - x_0, 0) - u \left( 0, t_0 - \frac{x_0}{c} \right) \right] = 0
\]

In this exercise $u(x, 0) = x$ and $u(0, t) = t^2$, so $u(x_0 + ct_0, 0) = x_0 + ct_0$ and $u(ct_0 - x_0, 0) = ct_0 - x_0$ and $u(0, t_0 - x_0/c) = (t_0 - x_0/c)^2$.

\[
-2cu(x_0, t_0) + 2c \left( t_0 - \frac{x_0}{c} \right)^2 + c[(x_0 + ct_0) - (ct_0 - x_0)] = 0
\]
Solve this equation for $2cu(x_0, t_0)$.

$$2cu(x_0, t_0) = 2c \left( t_0 - \frac{x_0}{c} \right)^2 + c[(x_0 + ct_0) - (ct_0 - x_0)]$$

Divide both sides by $2c$.

$$u(x_0, t_0) = \left( t_0 - \frac{x_0}{c} \right)^2 + \frac{1}{2}[(x_0 + ct_0) - (ct_0 - x_0)]$$

$$= \left( t_0 - \frac{x_0}{c} \right)^2 + \frac{1}{2}(2x_0)$$

$$= x_0 + \left( t_0 - \frac{x_0}{c} \right)^2$$

Therefore, switching the roles of $x$ and $t$ with those of $x_0$ and $t_0$, respectively,

$$u(x, t) = x + \left( t - \frac{x}{c} \right)^2, \quad x - ct < 0.$$ 

In conclusion, the solution to the initial boundary value problem is

$$u(x, t) = \begin{cases} 
  x + \left( t - \frac{x}{c} \right)^2 & \text{if } x - ct < 0 \\
  x & \text{if } x - ct > 0
\end{cases}.$$