Exercise 14

Solve the homogeneous wave equation on the half-line $(0, \infty)$ with zero initial data and with the Neumann boundary condition $u_x(0, t) = k(t)$. Use any method you wish.

Solution

The initial boundary value problem to solve is as follows.

$$\begin{gathered} u_{tt} = c^2 u_{xx}, \quad 0 < x < \infty,\ t > 0 \\ u(x, 0) = 0 \qquad u_t(x, 0) = 0 \\ u_x(0, t) = k(t) \end{gathered}$$

Because of the Neumann boundary condition, Green's theorem will not be used to solve this problem. Rather, we will use the method of reflection since the wave equation is over a semi-infinite interval $0 < x < \infty$. In order to make the Neumann boundary condition homogeneous, make the change of variables,

$$v(x, t) = u(x, t) - xk(t).$$

Find the derivatives of u in terms of this new variable.

$$\begin{aligned} u_t &= v_t + xk'(t) \\ u_{tt} &= v_{tt} + xk''(t) \\ u_x &= v_x + k(t) \\ u_{xx} &= v_{xx} \end{aligned}$$

As a result, the PDE v satisfies is

$$v_{tt} + xk''(t) = c^2 v_{xx},$$

or

$$v_{tt} - c^2 v_{xx} = -xk''(t).$$

The initial and boundary conditions associated with it are

$$\begin{aligned} v(x, 0) &= u(x, 0) - xk(0) = 0 - xk(0) = -xk(0) \\ v_t(x, 0) &= u_t(x, 0) - xk'(0) = 0 - xk'(0) = -xk'(0) \\ v_x(0, t) &= u_x(0, t) - k(t) = k(t) - k(t) = 0. \end{aligned}$$

Apply the method of reflection to solve for v. Consider the same problem over the whole line, using the even extension of ϕ in order to satisfy the Neumann boundary condition at $z = 0$:

$$\begin{gathered} V_{tt} - c^2 V_{xx} = f_{\text{even}}(x, t), \quad -\infty < x < \infty,\ t > 0 \\ V(x, 0) = \phi_{\text{even}}(x), \quad V_t(x, 0) = \psi_{\text{even}}(x), \end{gathered}$$

where

$$\phi_{\text{even}}(x) = \begin{cases} -xk(0) & \text{if } x > 0 \\ xk(0) & \text{if } x < 0 \end{cases} \quad \text{and} \quad \psi_{\text{even}}(x) = \begin{cases} -xk'(0) & \text{if } x > 0 \\ xk'(0) & \text{if } x < 0 \end{cases} \quad \text{and} \quad f_{\text{even}}(x, t) = \begin{cases} -xk''(t) & \text{if } x > 0 \\ xk''(t) & \text{if } x < 0 \end{cases}.$$

The solution for the inhomogeneous wave equation over the whole line is given by

$$V(x,t) = \frac{1}{2}[\phi_{\text{even}}(x+ct) + \phi_{\text{even}}(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi_{\text{even}}(x_0)\,dx_0 + \frac{1}{2c}\int_0^t \int_{x-c(t-t_0)}^{x+c(t-t_0)} f_{\text{even}}(x_0,t_0)\,dx_0\,dt_0.$$

The solution for v is then just the restriction of V to $x > 0$.

$$v(x,t) = \frac{1}{2}[\phi_{\text{even}}(x+ct) + \phi_{\text{even}}(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi_{\text{even}}(x_0)\,dx_0 + \frac{1}{2c}\int_0^t \int_{x-c(t-t_0)}^{x+c(t-t_0)} f_{\text{even}}(x_0,t_0)\,dx_0\,dt_0, \quad x > 0$$

Our task now is to write this formula in terms of the given function $k(t)$. Note that

$$\phi_{\text{even}}(x+ct) = \begin{cases} -(x+ct)k(0) & \text{if } x+ct > 0 \\ (x+ct)k(0) & \text{if } x+ct < 0 \end{cases} \quad \text{and} \quad \phi_{\text{even}}(x-ct) = \begin{cases} -(x-ct)k(0) & \text{if } x-ct > 0 \\ (x-ct)k(0) & \text{if } x-ct < 0 \end{cases},$$

so for every region in the xt-quarter-plane, we have to test whether $x - ct$ and $x + ct$ are greater than or less than zero.

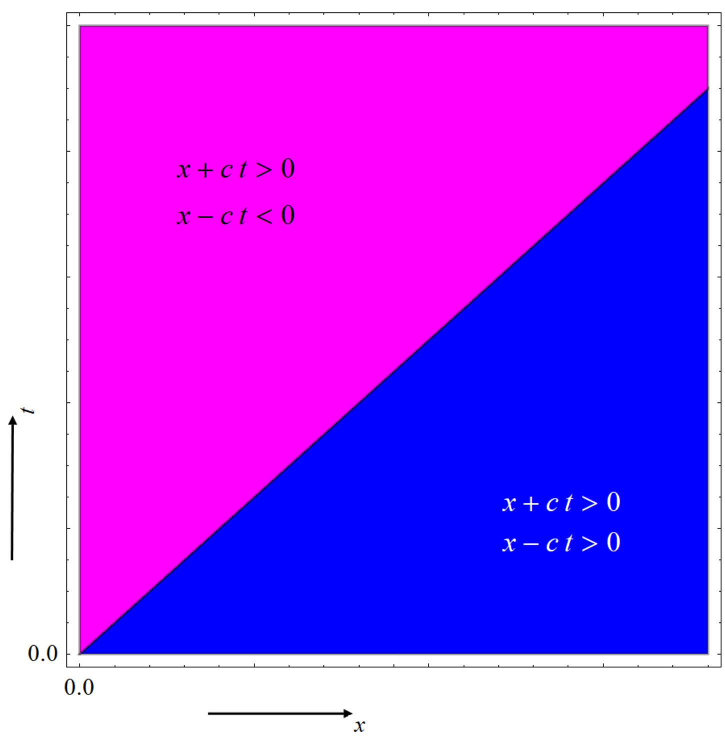

Figure 1: This figure illustrates the regions in the xt-quarter-plane that come about from using the even extension of each function. The solution for v has to be considered in each one. The characteristic line $x - ct = 0$ is the line that separates the regions.

The Blue Region: $x - ct > 0$

In the blue region $x + ct > 0$ and $x - ct > 0$, so the solution for v is

$$\begin{aligned}
v(x,t) &= \frac{1}{2}[\phi_{\text{even}}(x+ct) + \phi_{\text{even}}(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi_{\text{even}}(x_0)\,dx_0 + \frac{1}{2c}\int_0^t \int_{x-c(t-t_0)}^{x+c(t-t_0)} f_{\text{even}}(x_0,t_0)\,dx_0\,dt_0 \\
&= \frac{1}{2}[-(x+ct)k(0) - (x-ct)k(0)] + \frac{1}{2c}\int_{x-ct}^{x+ct} [-x_0 k'(0)]\,dx_0 + \frac{1}{2c}\int_0^t \int_{x-c(t-t_0)}^{x+c(t-t_0)} [-x_0 k''(t_0)]\,dx_0\,dt_0 \\
&= \frac{1}{2}[-2xk(0)] - \frac{k'(0)}{2c}\frac{x_0^2}{2}\bigg|_{x-ct}^{x+ct} - \frac{1}{2c}\int_0^t k''(t_0)\frac{x_0^2}{2}\bigg|_{x-c(t-t_0)}^{x+c(t-t_0)}\,dt_0 \\
&= -xk(0) - \frac{k'(0)}{2c}(2ctx) - \frac{1}{2c}\int_0^t k''(t_0)[2cx(t-t_0)]\,dt_0 \\
&= -xk(0) - txk'(0) + x\int_0^t k''(t_0)(t_0-t)\,dt_0 \\
&= -xk(0) - txk'(0) + x\left[k'(t_0)(t_0-t)\bigg|_0^t - \int_0^t k'(t_0)\,dt_0\right] \\
&= -xk(0) - txk'(0) + x[-k'(0)(-t) - k(t) + k(0)] \\
&= -xk(0) - txk'(0) + x[tk'(0) - k(t) + k(0)] \\
&= -\cancel{xk(0)} - \cancel{txk'(0)} + \cancel{txk'(0)} - xk(t) + \cancel{xk(0)} \\
&= -xk(t)
\end{aligned}$$

Therefore, since $u(x,t) = v(x,t) + xk(t)$,

$$u(x,t) = 0, \quad x - ct > 0.$$

The Magenta Region: $x - ct < 0$

In the magenta region $x + ct > 0$ and $x - ct < 0$, so the solution for v is

$$\begin{aligned}
v(x,t) &= \frac{1}{2}[\phi_{\text{even}}(x+ct) + \phi_{\text{even}}(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi_{\text{even}}(x_0)\,dx_0 + \frac{1}{2c}\int_0^t \int_{x-c(t-t_0)}^{x+c(t-t_0)} f_{\text{even}}(x_0,t_0)\,dx_0\,dt_0 \\
&= \frac{1}{2}[-(x+ct)k(0) + (x-ct)k(0)] + \frac{1}{2c}\left[\int_{x-ct}^{0}[x_0 k'(0)]\,dx_0 + \int_0^{x+ct}[-x_0 k'(0)]\,dx_0\right] \\
&\quad + \frac{1}{2c}\left\{\int_0^{t-\frac{x}{c}}\left[\int_{x-c(t-t_0)}^{0}[x_0 k''(t_0)]\,dx_0 + \int_0^{x+c(t-t_0)}[-x_0 k''(t_0)]\,dx_0\right]dt_0\right. \\
&\qquad\qquad \left. + \int_{t-\frac{x}{c}}^{t}\int_{x-c(t-t_0)}^{x+c(t-t_0)}[-x_0 k''(t_0)]\,dx_0\,dt_0\right\} \\
&= \frac{1}{2}[-2ctk(0)] + \frac{k'(0)}{2c}\left(\frac{x_0^2}{2}\bigg|_{x-ct}^{0} - \frac{x_0^2}{2}\bigg|_0^{x+ct}\right) \\
&\quad + \frac{1}{2c}\left\{\int_0^{t-\frac{x}{c}} k''(t_0)\left[\frac{x_0^2}{2}\bigg|_{x-c(t-t_0)}^{0} - \frac{x_0^2}{2}\bigg|_0^{x+c(t-t_0)}\right]dt_0 - \int_{t-\frac{x}{c}}^{t} k''(t_0)\,\frac{x_0^2}{2}\bigg|_{x-c(t-t_0)}^{x+c(t-t_0)}\,dt_0\right\} \\
&= -ctk(0) + \frac{k'(0)}{4c}[-(x-ct)^2 - (x+ct)^2] + \frac{1}{4c}\left\{\int_0^{t-\frac{x}{c}} k''(t_0)[-(x-ct+ct_0)^2 - (x+ct-ct_0)^2]\,dt_0\right. \\
&\qquad\qquad \left. - \int_{t-\frac{x}{c}}^{t} k''(t_0)[(x+ct-ct_0)^2 - (x-ct+ct_0)^2]\,dt_0\right\} \\
&= -ctk(0) + \frac{k'(0)}{4c}[-2(x^2+c^2t^2)] + \frac{1}{4c}\left\{\int_0^{t-\frac{x}{c}} k''(t_0)\{-2[x^2 + c^2(t_0-t)^2]\}\,dt_0\right. \\
&\qquad\qquad \left. - \int_{t-\frac{x}{c}}^{t} k''(t_0)[-4cx(t_0-t)]\,dt_0\right\} \\
&= -ctk(0) - \frac{k'(0)}{2c}(x^2+c^2t^2) - \frac{1}{2c}\int_0^{t-\frac{x}{c}} k''(t_0)[x^2 + c^2(t_0-t)^2]\,dt_0 + x\int_{t-\frac{x}{c}}^{t} k''(t_0)(t_0-t)\,dt_0 \\
&= -ctk(0) - \frac{k'(0)}{2c}(x^2+c^2t^2) \\
&\quad - \frac{1}{2c}\left[k'(t_0)[x^2 + c^2(t_0-t)^2]\bigg|_0^{t-\frac{x}{c}} - \int_0^{t-\frac{x}{c}} k'(t_0)[2c^2(t_0-t)]\,dt_0\right] \\
&\qquad + x\left[k'(t_0)(t_0-t)\bigg|_{t-\frac{x}{c}}^{t} - \int_{t-\frac{x}{c}}^{t} k'(t_0)\,dt_0\right] \\
&= -ctk(0) - \frac{k'(0)}{2c}(x^2+c^2t^2) \\
&\quad - \frac{1}{2c}\left[k'\left(t-\frac{x}{c}\right)(2x^2) - k'(0)(x^2+c^2t^2) - 2c^2\int_0^{t-\frac{x}{c}} k'(t_0)(t_0-t)\,dt_0\right] \\
&\qquad + x\left[-k'\left(t-\frac{x}{c}\right)\left(-\frac{x}{c}\right) - k(t) + k\left(t-\frac{x}{c}\right)\right].
\end{aligned}$$

Continue with the simplification.

$$\begin{aligned} v(x,t) &= -ctk(0) - \cancel{\frac{k'(0)}{2c}(x^2+c^2t^2)} - \cancel{\frac{x^2}{c}k'\left(t-\frac{x}{c}\right)} + \cancel{\frac{k'(0)}{2c}(x^2+c^2t^2)} + c\int_0^{t-\frac{x}{c}} k'(t_0)(t_0-t)\,dt_0 \\ &\quad + \cancel{\frac{x^2}{c}k'\left(t-\frac{x}{c}\right)} - xk(t) + xk\left(t-\frac{x}{c}\right) \\ &= -ctk(0) - xk(t) + xk\left(t-\frac{x}{c}\right) + c\int_0^{t-\frac{x}{c}} k'(t_0)(t_0-t)\,dt_0 \\ &= -ctk(0) - xk(t) + xk\left(t-\frac{x}{c}\right) + c\left[k(t_0)(t_0-t)\Big|_0^{t-\frac{x}{c}} - \int_0^{t-\frac{x}{c}} k(t_0)\,dt_0\right] \\ &= -ctk(0) - xk(t) + xk\left(t-\frac{x}{c}\right) + c\left[k\left(t-\frac{x}{c}\right)\left(-\frac{x}{c}\right) - k(0)(-t) - \int_0^{t-\frac{x}{c}} k(t_0)\,dt_0\right] \\ &= -\cancel{ctk(0)} - xk(t) + \cancel{xk\left(t-\frac{x}{c}\right)} - \cancel{xk\left(t-\frac{x}{c}\right)} + \cancel{ctk(0)} - c\int_0^{t-\frac{x}{c}} k(t_0)\,dt_0 \\ &= -xk(t) - c\int_0^{t-\frac{x}{c}} k(t_0)\,dt_0 \end{aligned}$$

Therefore, since $u(x,t) = v(x,t) + xk(t)$,

$$u(x,t) = -c\int_0^{t-\frac{x}{c}} k(t_0)\,dt_0, \quad x - ct < 0.$$

In conclusion, the solution to the initial boundary value problem is

$$u(x,t) = \begin{cases} -c\displaystyle\int_0^{t-\frac{x}{c}} k(t_0)\,dt_0 & \text{if } x-ct<0 \\ 0 & \text{if } x-ct>0 \end{cases}.$$