Exercise 5

Let \( f(x, t) \) be any function and let \( u(x, t) = (1/2c) \int_{\Delta} f \), where \( \Delta \) is the triangle of dependence. Verify directly by differentiation that

\[
    u_{tt} = c^2 u_{xx} + f \quad \text{and} \quad u(x, 0) \equiv u_t(x, 0) \equiv 0.
\]

(Hint: Begin by writing the formula as the iterated integral

\[
    u(x, t) = \frac{1}{2c} \int_0^t \int_{x-ct+cs}^{x+ct-cs} f(y, s) \, dy \, ds
\]

and differentiate with care using the rule in the Appendix. This exercise is not easy.)

Solution

The aim here is to verify that

\[
    u(x, t) = \frac{1}{2c} \int_0^t \int_{x-ct+cs}^{x+ct-cs} f(y, s) \, dy \, ds
\]

satisfies the inhomogeneous wave equation, \( u_{tt} = c^2 u_{xx} + f(x, t) \). In order to differentiate this double integral, it’s necessary to use the Leibnitz rule, which states that if

\[
    I(t) = \int_{a(t)}^{b(t)} \gamma(x, t) \, dx,
\]

then

\[
    \frac{dI}{dt} = \int_{a(t)}^{b(t)} \frac{\partial \gamma}{\partial t} \, dx + \gamma(b(t), t)b'(t) - \gamma(a(t), t)a'(t).
\]

Apply the rule twice to obtain the first derivative of \( u \) with respect to \( t \).

\[
    \frac{\partial u}{\partial t} = \frac{1}{2c} \frac{\partial}{\partial t} \int_0^t \int_{x-ct+cs}^{x+ct-cs} f(y, s) \, dy \, ds
\]

\[
    = \frac{1}{2c} \left\{ \int_0^t \left[ \frac{\partial}{\partial t} \int_{x-c(t-s)}^{x+c(t-s)} f(y, s) \, dy \right] ds + \int_x^x f(y, t) \, dy \times 1 - \int_{x-ct}^{x+ct} f(y, 0) \, dy \times 0 \right\}
\]

\[
    = \frac{1}{2c} \int_0^t \left[ \frac{\partial}{\partial t} \int_{x-c(t-s)}^{x+c(t-s)} f(y, s) \, dy \right] ds
\]

\[
    = \frac{1}{2c} \int_0^t \left\{ \int_{x-c(t-s)}^{x-c(t-s)} \frac{\partial}{\partial t} f(y, s) \, dy + f[x + c(t-s), s] \times (c) - f[x - c(t-s), s] \times (-c) \right\} ds
\]

Thus,

\[
    \frac{\partial u}{\partial t} = \frac{1}{2} \int_0^t \{ f[x + c(t-s), s] + f[x - c(t-s), s] \} \, ds.
\]
Thus, $u$ will be obtained.

Apply the rule once more to obtain the second derivative of $u$ with respect to $t$.

$$\frac{\partial^2 u}{\partial t^2} = \frac{1}{2} \frac{\partial}{\partial t} \int_0^t \{ f[x + c(t-t), s] + f[x + c(t-s), s] \} \, ds$$

$$= \frac{1}{2} \left\{ \int_0^t \frac{\partial}{\partial t} \{ f[x + c(t-t), s] + f[x + c(t-s), s] \} \, ds \right\}$$

$$+ [f(x, t) + f(x, t)] \times 1 - [f(x + ct, 0) + f(x - ct, 0)] \times 0$$

$$= \frac{1}{2} \int_0^t \{ f_a(x + c(t-t), s) \times (c) + f_a(x - c(t-s), s) \times (-c) \} \, ds + f(x, t),$$

where $f_a$ is a derivative of $f$ with respect to its first argument. Thus,

$$\frac{\partial^2 u}{\partial t^2} = \frac{c}{2} \int_0^t \{ f_a[x + c(t-t), s] - f_a[x - c(t-s), s] \} \, ds + f(x, t).$$

Now we will take derivatives of $u$ with respect to $x$. We have the following for the first derivative.

$$\frac{\partial u}{\partial x} = \frac{1}{2c} \frac{\partial}{\partial x} \int_0^t \int_{-c(t-t)}^{c(t-c)} f(y, s) \, dy \, ds$$

$$= \frac{1}{2c} \int_0^t \left[ \frac{\partial}{\partial x} \int_{-c(t-t)}^{c(t-c)} f(y, s) \, dy \right] \, ds$$

$$= \frac{1}{2c} \int_0^t \left\{ \int_{-c(t-t)}^{c(t-c)} f(y, s) \, dy + f[x + c(t-t), s] \times 1 - f[x - c(t-s), s] \times 1 \right\} \, ds$$

Thus,

$$\frac{\partial u}{\partial x} = \frac{1}{2c} \int_0^t \{ f[x + c(t-t), s] - f[x - c(t-s), s] \} \, ds.$$

Now the second derivative of $u$ with respect to $x$ will be obtained.

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{2c} \frac{\partial}{\partial x} \int_0^t \{ f[x + c(t-t), s] - f[x - c(t-s), s] \} \, ds$$

$$= \frac{1}{2c} \int_0^t \frac{\partial}{\partial x} \{ f[x + c(t-t), s] - f[x - c(t-s), s] \} \, ds$$

$$= \frac{1}{2c} \int_0^t \{ f_a[x + c(t-t), s] \times 1 - f_a[x - c(t-s), s] \times 1 \} \, ds,$$

where $f_a$ is a derivative of $f$ with respect to its first argument. Thus,

$$\frac{\partial^2 u}{\partial x^2} = \frac{c}{2} \frac{\partial^2 u}{\partial t^2} + f(x, t)$$

Therefore,

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} + f(x, t)$$

and

$$u(x, 0) = \frac{1}{2c} \int_0^0 \int_{x-c}^{x+c} f(y, s) \, dy \, ds = 0 \quad \text{and} \quad u_t(x, 0) = \frac{1}{2} \int_0^0 \{ f[x + c(t-t), s] + f[x - c(t-s), s] \} \, ds = 0.$$

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