Exercise 6

Derive the formula for the inhomogeneous wave equation in yet another way.

(a) Write it as the system
\[ u_t + cu_x = v, \quad v_t - cv_x = f. \]

(b) Solve the first equation for \( u \) in terms of \( v \) as
\[ u(x, t) = \int_0^t v(x - ct + cs, s) \, ds. \]

(c) Similarly, solve the second equation for \( v \) in terms of \( f \).

(d) Substitute part (c) into part (b) and write as an iterated integral.

Solution

The PDE we have to solve is
\[ u_{tt} = c^2 u_{xx} + f(x, t) \]
over the whole line. There are two initial conditions,
\[ u(x, 0) = \phi(x) \quad \text{and} \quad u_t(x, 0) = \psi(x). \]

Part (a)

Bring \( c^2 u_{xx} \) to the other side.
\[ u_{tt} - c^2 u_{xx} = f(x, t) \]

Write the left side as an operator acting on \( u \).
\[ (\partial_t^2 - c^2 \partial_x^2)u = f(x, t) \]

The operator is a difference of squares, so it can be factored.
\[ (\partial_t - c \partial_x)(\partial_t + c \partial_x)u = f(x, t) \]

Let
\[ v = (\partial_t + c \partial_x)u \]

so that the PDE becomes
\[ (\partial_t - c \partial_x)v = f(x, t). \]

The second-order PDE we started with has thus been reduced to the following system of first-order PDEs that can be solved with the method of characteristics.

\[ u_t + cu_x = v \quad \text{(1)} \]
\[ v_t - cv_x = f(x, t) \quad \text{(2)} \]
Part (b)

For a function of two variables \( z = z(x,t) \), its differential is defined as
\[
 dz = \frac{\partial z}{\partial t} \, dt + \frac{\partial z}{\partial x} \, dx. 
\]
If we divide both sides by \( dt \), then we get the relationship between the ordinary derivative of \( \phi \) and its partial derivatives.
\[
\frac{dz}{dt} = \frac{\partial z}{\partial t} + \frac{\partial z}{\partial x} \frac{dx}{dt} \tag{3}
\]
Comparing this with equation (1), we see that along the curves in the \( xt \)-plane that satisfy
\[
\frac{dx}{dt} = c, \tag{4}
\]
the PDE for \( u(x,t) \) reduces to an ODE.
\[
\frac{du}{dt} = v(x,t) \tag{5}
\]
Because \( c \) is a constant, equation (4) can be solved by integrating both sides with respect to \( t \).
\[
x = ct + \xi, \tag{6}
\]
where \( \xi \) is a characteristic coordinate. Substitute this expression for \( x \) into equation (5) to obtain an ODE that only involves \( t \) (\( \xi \) is regarded as a constant).
\[
\frac{du}{dt} = v(ct + \xi, t) 
\]
Integrate both sides with respect to \( t \).
\[
u(\xi,t) = \int_0^t v(cr + \xi, r) \, dr + g(\xi),
\]
where \( g \) is an arbitrary function of \( \xi \). The lower limit of integration is arbitrary and has been set equal to 0. In order to change back to the original variable \( x \), solve equation (6) for \( \xi \).
\[
x = ct + \xi \quad \rightarrow \quad \xi = x - ct
\]
Therefore,
\[
u(x,t) = \int_0^t v(cr + x - ct, r) \, dr + g(x - ct).
\]
Part (c)

Comparing equation (3) with equation (2), we see that along the curves in the \( xt \)-plane that satisfy
\[
\frac{dx}{dt} = -c, \tag{7}
\]
the PDE for \( v(x,t) \) reduces to an ODE.
\[
\frac{dv}{dt} = f(x,t) \tag{8}
\]
Because $c$ is a constant, equation (7) can be solved by integrating both sides with respect to $t$.

$$x = -ct + \eta,$$

where $\eta$ is a characteristic coordinate. Substitute this expression for $x$ into equation (8) to obtain an ODE that only involves $t$ ($\eta$ is regarded as a constant).

$$\frac{dv}{dt} = f(-ct + \eta, t)$$

Integrate both sides with respect to $t$.

$$v(\eta, t) = \int_0^t f(-cs + \eta, s) \, ds + h(\eta),$$

where $h$ is an arbitrary function of $\eta$. The lower limit of integration is arbitrary and has been set equal to 0. In order to change back to the original variable $x$, solve equation (9) for $\eta$.

$$x = -ct + \eta \rightarrow \eta = x + ct$$

Therefore,

$$v(x, t) = \int_0^t f(-cs + x + ct, s) \, ds + h(x + ct).$$

**Part (d)**

Substitute the solution for $v(x, t)$ into the one for $u(x, t)$.

$$u(x, t) = \int_0^t v(cr + x - ct, r) \, dr + g(x - ct)$$

$$= \int_0^t \left\{ \int_0^r f[-cs + (cr + x - ct) + cr, s] \, ds + h[(cr + x - ct) + cr] \right\} \, dr + g(x - ct)$$

$$= \int_0^t \int_0^r f(x - ct + 2cr - cs, s) \, ds \, dr + \int_0^t h(x - ct + 2cr) \, dr + g(x - ct)$$

Make the following substitution in the single integral.

$$p = x - ct + 2cr$$

$$dp = 2c \, dr \rightarrow \frac{1}{2c} \, dp = dr$$

The formula for $u$ becomes

$$u(x, t) = \int_0^t \int_0^r f(x - ct + 2cr - cs, s) \, ds \, dr + \int_{x-ct}^{x+ct} h(p) \left( \frac{1}{2c} \, dp \right) + g(x - ct)$$

$$= \int_0^t \int_0^r f(x - ct + 2cr - cs, s) \, ds \, dr + \frac{1}{2c} \int_{x-ct}^{x+ct} h(p) \, dp + g(x - ct)$$

$$= \int_0^t \int_0^r f(x - ct + 2cr - cs, s) \, ds \, dr + \frac{1}{2c} H(x + ct) - \frac{1}{2c} H(x - ct) + g(x - ct).$$
Use new arbitrary functions, \( A(x - ct) \) and \( B(x + ct) \), to simplify the expression.

\[
u(x,t) = \int_0^t \int_0^r f(x - ct + 2cr - cs, s) \, ds \, dr + A(x - ct) + B(x + ct)
\]

In order to simplify the double integral we will switch the order of integration. At the moment, the inner integral is in \( ds \), and \( s \) is present in both of \( f \)'s arguments. \( r \), on the other hand, is only in the first argument, so we can simplify the integrand if we make the inner integral in \( dr \).

Figure 1: The current mode of integration in the \( sr \)-plane is shown on the left. This domain will be integrated over as shown on the right to simplify the integral.

\[
u(x,t) = \int_0^t \int_0^r f(x - ct + 2cr - cs, s) \, dr \, ds + A(x - ct) + B(x + ct)
\]

Now the following substitution can be made in the integral.

\[
y = x - ct + 2cr - cs \quad dy = 2c \, dr \quad \rightarrow \quad \frac{1}{2c} \, dy = dr
\]

The result is

\[
u(x,t) = \int_0^t \int_{x-ct+cs}^{x+ct-cs} f(y, s) \left( \frac{1}{2c} \, dy \right) \, ds + A(x - ct) + B(x + ct).
\]

Therefore,

\[
u(x,t) = A(x - ct) + B(x + ct) + \frac{1}{2c} \int_0^t \int_{x-c(t-s)}^{x+c(t-s)} f(y, s) \, dy \, ds.
\]

This is the general solution to \( u_{tt} = c^2 u_{xx} + f \). If we apply the two initial conditions, we can determine \( A \) and \( B \).
Before doing so, take a derivative of the solution with respect to $t$.

$$u_t(x, t) = -cA'(x - ct) + cB'(x + ct) + \frac{1}{2c} \frac{\partial}{\partial t} \int_0^t \int_{x-c(t-s)}^{x+c(t-s)} f(y, s) \, dy \, ds$$

$$= -cA'(x - ct) + cB'(x + ct) + \frac{1}{2c} \int_0^t \left[ \frac{\partial}{\partial t} \int_{x-c(t-s)}^{x+c(t-s)} f(y, s) \, dy \right] \, ds + \int_0^x f(y, t) \, dy$$

$$= -cA'(x - ct) + cB'(x + ct) + \frac{1}{2c} \int_0^t \left\{ \int_{x-c(t-s)}^{x+c(t-s)} \frac{\partial}{\partial t} f(y, s) \, dy + f[x + c(t - s), s] \times (c) - f[x - c(t - s), s] \times (-c) \right\} \, ds$$

In differentiating the double integral, I made use of the Leibnitz integration rule which states that if

$$I(t) = \int_{a(t)}^{b(t)} \gamma(x, t) \, dx,$$

then

$$\frac{dI}{dt} = \int_{a(t)}^{b(t)} \frac{\partial \gamma}{\partial t} \, dx + \gamma[b(t), t]b'(t) - \gamma[a(t), t]a'(t).$$

From the initial conditions we obtain the following system of equations.

$$u(x, 0) = A(x) + B(x) = \phi(x)$$

$$u_t(x, 0) = -cA'(x) + cB'(x) = \psi(x)$$

Even though this system is in terms of $x$, it’s really in terms of $w$, where $w$ is any expression we choose.

$$A(w) + B(w) = \phi(w)$$

$$-cA'(w) + cB'(w) = \psi(w)$$

Differentiating both sides of the first equation with respect to $w$, we get

$$A'(w) + B'(w) = \phi'(w) \quad \rightarrow \quad B'(w) = \phi'(w) - A'(w).$$

Plug this expression for $B'(w)$ into the second equation.

$$-cA'(w) + c[\phi'(w) - A'(w)] = \psi(w) \quad \rightarrow \quad -2cA'(w) + c\phi'(w) = \psi(w) \quad \rightarrow \quad A'(w) = \frac{1}{2} \phi'(w) - \frac{1}{2c} \psi(w).$$

Solve for $A(w)$ and obtain an expression for $A(x - ct)$.

$$A(w) = \frac{1}{2} \phi(w) - \int_0^w \frac{1}{2c} \psi(s) \, ds + C_1 \quad \Rightarrow \quad A(x - ct) = \frac{1}{2} \phi(x - ct) - \int_{x-ct}^x \frac{1}{2c} \psi(s) \, ds + C_1$$
Use the first equation to solve for $B(w)$ and obtain an expression for $B(x + ct)$.

\[
B(w) = \phi(w) - A(w) \\
= \phi(w) - \frac{1}{2} \phi(w) + \int_{w}^{w} \frac{1}{2c} \psi(s) \, ds - C_1 \\
= \frac{1}{2} \phi(w) + \int_{w}^{w} \frac{1}{2c} \psi(s) \, ds - C_1 \quad \Rightarrow \quad B(x + ct) = \frac{1}{2} \phi(x + ct) + \int_{x}^{x+ct} \frac{1}{2c} \psi(s) \, ds - C_1
\]

The general solution for $u(x, t)$ becomes

\[
u(x, t) = A(x - ct) + B(x + ct) + \frac{1}{2c} \int_{0}^{t} \int_{x-c(t-s)}^{x+c(t-s)} f(y, s) \, dy \, ds \\
= \frac{1}{2} \phi(x - ct) - \int_{x-ct}^{x-ct} \frac{1}{2c} \psi(s) \, ds + \phi(x - ct) + \frac{1}{2c} \int_{x}^{x+ct} \psi(s) \, ds - \phi(x) + \phi(x) + \frac{1}{2c} \int_{0}^{t} \int_{x-c(t-s)}^{x+c(t-s)} f(y, s) \, dy \, ds \\
= \frac{1}{2} [\phi(x + ct) + \phi(x - ct)] + \int_{x-ct}^{x+ct} \frac{1}{2c} \psi(s) \, ds + \frac{1}{2c} \int_{x}^{x+ct} \psi(s) \, ds + \frac{1}{2c} \int_{x}^{x+ct} \psi(s) \, ds + \frac{1}{2c} \int_{0}^{t} \int_{x-c(t-s)}^{x+c(t-s)} f(y, s) \, dy \, ds.
\]

Therefore,

\[
u(x, t) = \frac{1}{2} [\phi(x + ct) + \phi(x - ct)] + \frac{1}{2c} \int_{x-ct}^{x+ct} \psi(s) \, ds + \frac{1}{2c} \int_{x}^{x+ct} \psi(s) \, ds + \frac{1}{2c} \int_{x}^{x+ct} \psi(s) \, ds + \frac{1}{2c} \int_{0}^{t} \int_{x-c(t-s)}^{x+c(t-s)} f(y, s) \, dy \, ds.
\]