Exercise 9

Let $u(t) = \int_0^t \mathscr{S}(t-s)f(s)\,ds$. Using *only* Exercise 8, show that u solves the inhomogeneous wave equation with zero initial data.

Solution

We have the following results from Exercise 8.

$$\mathscr{S}_{tt} - c^2\mathscr{S}_{xx} = 0, \quad \mathscr{S}(0) = 0, \quad \mathscr{S}_t(0) = I$$

Differentiate u with respect to t using the Leibniz rule.

$$\begin{aligned}
\frac{\partial u}{\partial t} &= \frac{\partial}{\partial t}\int_0^t \mathscr{S}(t-s)f(s)\,ds \\
&= \int_0^t \frac{\partial}{\partial t}\mathscr{S}(t-s)f(s)\,ds + \mathscr{S}(0)f(t)\cdot 1 - \mathscr{S}(t)f(0)\cdot 0 \\
&= \int_0^t \mathscr{S}_t(t-s)f(s)\,ds + \mathscr{S}(0)f(t) \\
&= \int_0^t \mathscr{S}_t(t-s)f(s)\,ds + (0)f(t) \\
&= \int_0^t \mathscr{S}_t(t-s)f(s)\,ds
\end{aligned}$$

Differentiate u with respect to t once more.

$$\begin{aligned}
\frac{\partial^2 u}{\partial t^2} &= \frac{\partial}{\partial t}\int_0^t \mathscr{S}_t(t-s)f(s)\,ds \\
&= \int_0^t \frac{\partial}{\partial t}\mathscr{S}_t(t-s)f(s)\,ds + \mathscr{S}_t(0)f(t)\cdot 1 - \mathscr{S}(t)f(0)\cdot 0 \\
&= \int_0^t \mathscr{S}_{tt}(t-s)f(s)\,ds + \mathscr{S}_t(0)f(t) \\
&= \int_0^t \mathscr{S}_{tt}(t-s)f(s)\,ds + (I)f(t) \\
&= \int_0^t \mathscr{S}_{tt}(t-s)f(s)\,ds + f(t) \\
&= \int_0^t c^2\mathscr{S}_{xx}(t-s)f(s)\,ds + f(t) \\
&= c^2\frac{\partial^2}{\partial x^2}\int_0^t \mathscr{S}(t-s)f(s)\,ds + f(t) \\
&= c^2\frac{\partial^2 u}{\partial x^2} + f(t)
\end{aligned}$$

$$u(0) = \int_0^0 \mathscr{S}(-s)f(s)\,ds = 0 \qquad \left.\frac{\partial u}{\partial t}\right|_{t=0} = \int_0^0 \mathscr{S}_t(-s)f(s)\,ds = 0$$

Therefore, u solves the inhomogeneous wave equation with zero initial data.