Exercise 2

Consider a metal rod ($0 < x < l$), insulated along its sides but not at its ends, which is initially at temperature = 1. Suddenly both ends are plunged into a bath of temperature = 0. Write the differential equation, boundary conditions, and initial condition. Write the formula for the temperature $u(x,t)$ at later times. In this problem, *assume* the infinite series expansion

$$1 = \frac{4}{\pi}\left(\sin\frac{\pi x}{l} + \frac{1}{3}\sin\frac{3\pi x}{l} + \frac{1}{5}\sin\frac{5\pi x}{l} + \cdots\right)$$

Solution

The governing PDE is the one-dimensional heat equation.

$$u_t = k u_{xx}, \quad 0 < x < l$$

The boundary conditions for it are

$$\begin{aligned} u(0,t) &= 0 \\ u(l,t) &= 0, \end{aligned}$$

and the initial condition is

$$u(x,0) = 1.$$

Since the heat equation and its boundary conditions are linear and homogeneous, the method of separation of variables can be applied to solve it. Assume a product solution of the form, $u(x,t) = X(x)T(t)$, and plug it into the PDE

$$u_t = k u_{xx} \quad \rightarrow \quad XT' = kX''T$$

and the boundary conditions.

$$\begin{aligned} u(0,t) = X(0)T(t) = 0 \quad &\rightarrow \quad X(0) = 0 \\ u(l,t) = X(l)T(t) = 0 \quad &\rightarrow \quad X(l) = 0 \end{aligned}$$

Separate variables now.

$$\frac{T'}{kT} = \frac{X''}{X}$$

Note that k is a constant and can go on either side. The final answer will be the same regardless. We have a function of t on the left side and a function of x on the right side. The only way both functions can be equal is if they are equal to a constant.

$$\frac{T'}{kT} = \frac{X''}{X} = p$$

Values of p for which $X(0) = 0$ and $X(l) = 0$ are satisfied are called the eigenvalues, and the nontrivial functions $X(x)$ associated with them are called the eigenfunctions.

Determination of Positive Eigenvalues: $p = \mu^2$

Assuming p is positive, the differential equation for X becomes

$$\frac{X''}{X} = \mu^2.$$

Multiply both sides by X.

$$X'' = \mu^2 X$$

The general solution can be written in terms of hyperbolic sine and hyperbolic cosine.

$$X(x) = C_1 \cosh \mu x + C_2 \sinh \mu x$$

Now use the boundary conditions to determine C_1 and C_2.

$$\begin{aligned} X(0) &= C_1 = 0 \\ X(l) &= C_1 \cosh \mu l + C_2 \sinh \mu l = 0 \end{aligned}$$

We see that $C_1 = 0$ and $C_2 = 0$. Hence, only the trivial solution $X(x) = 0$ results from considering positive values for p, and there are no positive eigenvalues.

Determination of the Zero Eigenvalue: $p = 0$

Assuming p is zero, the differential equation for X becomes

$$\frac{X''}{X} = 0.$$

Multiply both sides by X.

$$X'' = 0$$

The general solution is a linear function.

$$X(x) = C_3 x + C_4$$

Now use the boundary conditions to determine C_3 and C_4.

$$\begin{aligned} X(0) &= C_4 = 0 \\ X(l) &= C_3 l + C_4 = 0 \end{aligned}$$

We see that $C_3 = 0$ and $C_4 = 0$. Hence, only the trivial solution $X(x) = 0$ results from considering $p = 0$, and zero is not an eigenvalue.

Determination of Negative Eigenvalues: $p = -\lambda^2$

Assuming p is negative, the differential equation for X becomes

$$\frac{X''}{X} = -\lambda^2.$$

Multiply both sides by X.

$$X'' = -\lambda^2 X$$

The general solution can be written in terms of sine and cosine.

$$X(x) = C_5 \cos \lambda x + C_6 \sin \lambda x$$

Now use the boundary conditions to determine C_5 and C_6.

$$X(0) = C_5 = 0$$
$$X(l) = C_5 \cos \lambda l + C_6 \sin \lambda l = 0$$

The second equation simplifies to $C_6 \sin \lambda l = 0$. To avoid getting the trivial solution, we insist that $C_6 \neq 0$. Doing so yields an equation for the eigenvalues.

$$\sin \lambda l = 0$$

Solve for λl.

$$\lambda l = n\pi, \quad n = 1, 2, \ldots$$

So then

$$\lambda = \lambda_n = \frac{n\pi}{l}, \quad n = 1, 2, \ldots.$$

The eigenfunctions associated with these eigenvalues are

$$X(x) = C_6 \sin \lambda x \quad \rightarrow \quad X_n(x) = \sin \lambda_n x, \quad n = 1, 2, \ldots.$$

Now solve the differential equation for $T(t)$.

$$\frac{T'}{kT} = -\lambda^2$$

Multiply both sides by k.

$$\frac{T'}{T} = -k\lambda^2$$

The left side is just the derivative of $\ln T$.

$$\frac{d}{dt}(\ln T) = -k\lambda^2$$

Integrate both sides with respect to t.

$$\ln T = -k\lambda^2 t + C_7$$

Exponentiate both sides.

$$T(t) = e^{-k\lambda^2 t + C_7} = e^{-k\lambda^2 t} e^{C_7}$$

Use a new constant of integration.

$$T(t) = C_8 e^{-k\lambda^2 t} \quad \rightarrow \quad T_n(t) = e^{-k\lambda_n^2 t}$$

According to the principle of linear superposition, the solution to the PDE for $u(x,t)$ is a linear combination of all products $T_n(t)X_n(x)$ over all the eigenvalues.

$$\begin{aligned}
u(x,t) &= \sum_{n=1}^{\infty} A_n T_n(t) X_n(x) \\
&= \sum_{n=1}^{\infty} A_n e^{-k\lambda_n^2 t} \sin \lambda_n x \\
&= \sum_{n=1}^{\infty} A_n e^{-k\left(\frac{n\pi}{l}\right)^2 t} \sin \frac{n\pi x}{l} \\
&= \sum_{n=1}^{\infty} A_n e^{-k\frac{n^2\pi^2}{l^2} t} \sin \frac{n\pi x}{l}
\end{aligned}$$

The final task is to use Fourier's method and the initial condition to determine A_n.

$$u(x,0) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{l} = 1$$

Multiply both sides by $\sin \lambda_m x$, where m is a positive integer.

$$u(x,0) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{l} \sin \frac{m\pi x}{l} = \sin \frac{m\pi x}{l}$$

Integrate both sides with respect to x over the domain the PDE is defined.

$$\int_0^l \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{l} \sin \frac{m\pi x}{l}\, dx = \int_0^l \sin \frac{m\pi x}{l}\, dx$$

Bring the integral inside the sum on the left and evaluate the integral on the right.

$$\sum_{n=1}^{\infty} A_n \int_0^l \sin \frac{n\pi x}{l} \sin \frac{m\pi x}{l}\, dx = -\frac{l}{m\pi} \cos \frac{m\pi x}{l}\bigg|_0^l$$

Since n and m are integers,

$$\int_0^l \sin \frac{n\pi x}{l} \sin \frac{m\pi x}{l}\, dx = \begin{cases} \frac{l}{2} & n = m \\ 0 & n \neq m \end{cases},$$

as can be verified with trigonometric identities. This implies that every term in the infinite series vanishes except for the $n = m$ term.

$$\begin{aligned}
A_n \cdot \frac{l}{2} &= -\frac{l}{n\pi}(\cos n\pi - 1) \\
A_n \cdot \frac{l}{2} &= -\frac{l}{n\pi}[(-1)^n - 1] \\
A_n \cdot \frac{l}{2} &= \frac{l}{n\pi}[-(-1)^n + 1] \\
A_n \cdot \frac{l}{2} &= \frac{l}{n\pi}[(-1)^{n+1} + 1]
\end{aligned}$$

The coefficient is

$$A_n = \frac{2}{n\pi}[(-1)^{n+1} + 1],$$

so

$$u(x,t) = \sum_{n=1}^{\infty} \frac{2}{n\pi}[(-1)^{n+1} + 1]e^{-k\frac{n^2\pi^2}{l^2}t} \sin\frac{n\pi x}{l}.$$

This answer for u is acceptable, but it can be simplified. If n is even, then the term in square brackets evaluates to 0. If n is odd, then the term in square brackets evaluates to 2.

$$A_n = \begin{cases} 0 & n \text{ is even} \\ \frac{4}{n\pi} & n \text{ is odd} \end{cases}$$

By replacing n with $2m - 1$ the series can be made to sum over the odd integers only. Therefore,

$$u(x,t) = \sum_{m=1}^{\infty} \frac{4}{(2m-1)\pi} e^{-k\frac{(2m-1)^2\pi^2}{l^2}t} \sin\frac{(2m-1)\pi x}{l}.$$

The advantage of using this form is that we don't need as many terms in the series to plot the solution.

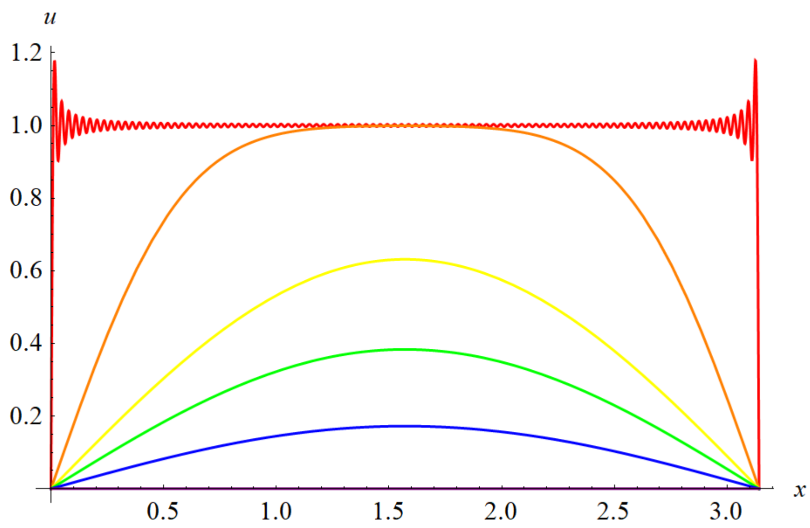

Figure 1: This is a plot of the temperature u in the rod as a function of x for $k = 1$ and $l = \pi$ at various times. The first 100 terms of the series were used. The curves in red, orange, yellow, green, blue, and purple correspond to $t = 0$, $t = 0.1$, $t = 0.7$, $t = 1.2$, $t = 2$, and $t = 10$, respectively.

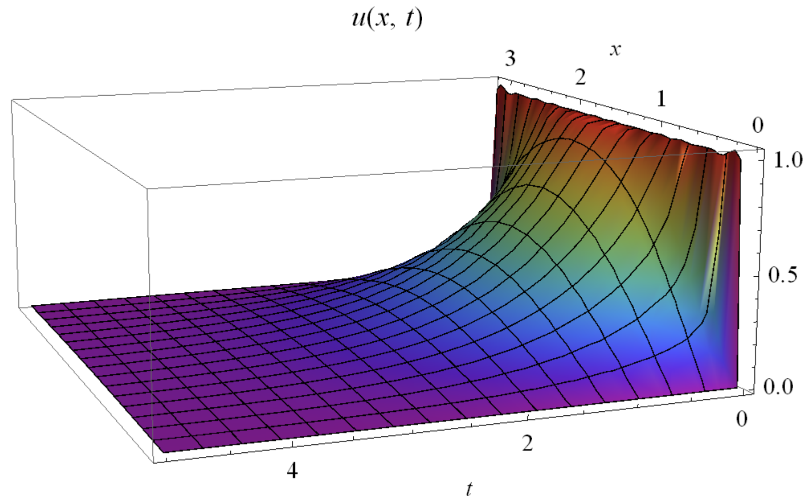

Figure 2: This is a plot of the two-dimensional solution surface $u(x,t)$ in three-dimensional space for $k = 1$, $l = \pi$, $0 < x < l$, and $0 < t < 5$. The first 100 terms of the series were used.