Exercise 5

Do the same for $2\pi c/l < r < 4\pi c/l$.

Solution

The PDE we have to solve is

$$u_{tt} = c^2 u_{xx} - r u_t \quad \text{for } 0 < x < l$$

subject to the boundary conditions,

$$u(0,t) = 0 \quad \text{and} \quad u(l,t) = 0,$$

and the initial conditions,

$$u(x,0) = \phi(x) \quad \text{and} \quad u_t(x,0) = \psi(x).$$

Because the PDE and its boundary conditions are linear and homogeneous, the method of separation of variables can be applied to solve it. Assume a product solution of the form, $u(x,t) = X(x)T(t)$, and plug it into the PDE

$$u_{tt} = c^2 u_{xx} - r u_t \quad \rightarrow \quad XT'' = c^2 X''T - rXT'$$

and the boundary conditions.

$$\begin{aligned} u(0,t) = X(0)T(t) = 0 \qquad &\rightarrow \qquad X(0) = 0 \\ u(l,t) = X(l)T(t) = 0 \qquad &\rightarrow \qquad X(l) = 0 \end{aligned}$$

Separate variables now.

$$XT'' + rXT' = c^2 X''T \quad \rightarrow \quad \frac{T'' + rT'}{c^2 T} = \frac{X''}{X}$$

Note that c^2 is a constant and can go on either side. The final answer will be the same regardless. We have a function of t on the left side and a function of x on the right side. The only way both functions can be equal is if they are equal to a constant.

$$\frac{T'' + rT'}{c^2 T} = \frac{X''}{X} = k$$

Values of k for which $X(0) = 0$ and $X(l) = 0$ are satisfied are called the eigenvalues, and the nontrivial functions $X(x)$ associated with them are called the eigenfunctions.

Determination of Positive Eigenvalues: $k = \mu^2$

Assuming k is positive, the differential equation for X becomes

$$\frac{X''}{X} = \mu^2$$

Multiply both sides by X.

$$X'' = \mu^2 X$$

The general solution can be written in terms of hyperbolic sine and hyperbolic cosine.

$$X(x) = C_1 \cosh \mu x + C_2 \sinh \mu x$$

Now use the boundary conditions to determine C_1 and C_2.

$$X(0) = C_1 = 0$$
$$X(l) = C_1 \cosh \mu l + C_2 \sinh \mu l = 0$$

We see that $C_1 = 0$ and $C_2 = 0$. Hence, only the trivial solution $X(x) = 0$ results from considering positive values for k, and there are no positive eigenvalues.

Determination of the Zero Eigenvalue: $k = 0$

Assuming k is zero, the differential equation for X becomes

$$\frac{X''}{X} = 0.$$

Multiply both sides by X.

$$X'' = 0$$

The general solution is a linear function.

$$X(x) = C_3 x + C_4$$

Now use the boundary conditions to determine C_3 and C_4.

$$X(0) = C_4 = 0$$
$$X(l) = C_3 l + C_4 = 0$$

We see that $C_3 = 0$ and $C_4 = 0$. Hence, only the trivial solution $X(x) = 0$ results from considering $k = 0$, and zero is not an eigenvalue.

Determination of Negative Eigenvalues: $k = -\lambda^2$

Assuming k is negative, the differential equation for X becomes

$$\frac{X''}{X} = -\lambda^2.$$

Multiply both sides by X.

$$X'' = -\lambda^2 X$$

The general solution can be written in terms of sine and cosine.

$$X(x) = C_5 \cos \lambda x + C_6 \sin \lambda x$$

Now use the boundary conditions to determine C_5 and C_6.

$$X(0) = C_5 = 0$$
$$X(l) = C_5 \cos \lambda l + C_6 \sin \lambda l = 0$$

The second equation simplifies to $C_6 \sin \lambda l = 0$. To avoid getting the trivial solution, we insist that $C_6 \neq 0$. Doing so yields an equation for the eigenvalues.

$$\sin \lambda l = 0$$

Solve for λl.

$$\lambda l = n\pi, \quad n = 1, 2, \ldots$$

So then

$$\lambda = \lambda_n = \frac{n\pi}{l}, \quad n = 1, 2, \ldots.$$

The eigenfunctions associated with these eigenvalues are

$$X(x) = C_6 \sin \lambda x \quad \rightarrow \quad X_n(x) = \sin \lambda_n x, \quad n = 1, 2, \ldots.$$

Now solve the differential equation for $T(t)$.

$$\frac{T'' + rT'}{c^2 T} = -\lambda^2$$

Multiply both sides by $c^2 T$.

$$T'' + rT' = -c^2\lambda^2 T$$

Bring $c^2\lambda^2 T$ to the left side.

$$T'' + rT' + c^2\lambda^2 T = 0$$

This is an ODE with constant coefficients, so its solution is of the form

$$T = e^{st}.$$

Substitute this into the ODE to determine s.

$$s^2 e^{st} + rse^{st} + c^2\lambda^2 e^{st} = 0$$

Divide both sides by e^{st}.

$$s^2 + rs + c^2\lambda^2 = 0$$

This is a quadratic equation for s, so use the quadratic formula to solve for it.

$$s = \frac{-r \pm \sqrt{r^2 - 4c^2\lambda^2}}{2}$$

Now substitute $\lambda = n\pi/l$.

$$s = \frac{-r \pm \sqrt{r^2 - 4c^2\left(\frac{n\pi}{l}\right)^2}}{2}$$

$$s = \frac{-r \pm \sqrt{r^2 - n^2\left(\frac{2\pi c}{l}\right)^2}}{2}$$

Since $2\pi c/l < r < 4\pi c/l$, the quantity under the square root is positive when $n = 1$ and negative when $n \geq 2$. Hence, there are two cases to consider.

$$s = \begin{cases} \dfrac{-r \pm \sqrt{r^2 - \left(\frac{2\pi c}{l}\right)^2}}{2} & n = 1 \\ \dfrac{-r \pm \sqrt{r^2 - n^2\left(\frac{2\pi c}{l}\right)^2}}{2} & n \geq 2 \end{cases}$$

In the second expression factor out -1 and bring it out of the square root as i. Now the square roots yield real numbers in both cases.

$$s = \begin{cases} \dfrac{-r \pm \sqrt{r^2 - \left(\frac{2\pi c}{l}\right)^2}}{2} & n = 1 \\ \dfrac{-r \pm i\sqrt{n^2\left(\frac{2\pi c}{l}\right)^2 - r^2}}{2} & n \geq 2 \end{cases}$$

$$s = \begin{cases} -\dfrac{r}{2} \pm \sqrt{\dfrac{r^2}{4} - \left(\dfrac{\pi c}{l}\right)^2} & n = 1 \\ -\dfrac{r}{2} \pm i\sqrt{n^2\left(\dfrac{\pi c}{l}\right)^2 - \dfrac{r^2}{4}} & n \geq 2 \end{cases}$$

$$s = \begin{cases} -\dfrac{r}{2} \pm \dfrac{\sqrt{r^2l^2 - 4\pi^2c^2}}{2l} & n = 1 \\ -\dfrac{r}{2} \pm i\dfrac{\sqrt{4n^2\pi^2c^2 - r^2l^2}}{2l} & n \geq 2 \end{cases}$$

Thus, the general solution to the ODE for T is

$$T(t) = \begin{cases} C_7 \exp\left[\left(-\dfrac{r}{2} + \dfrac{\sqrt{r^2l^2 - 4\pi^2c^2}}{2l}\right)t\right] + C_8 \exp\left[\left(-\dfrac{r}{2} - \dfrac{\sqrt{r^2l^2 - 4\pi^2c^2}}{2l}\right)t\right] & n = 1 \\ C_9 \exp\left(-\dfrac{r}{2}t\right)\cos\left(\dfrac{\sqrt{4n^2\pi^2c^2 - r^2l^2}}{2l}t\right) + C_{10}\exp\left(-\dfrac{r}{2}t\right)\sin\left(\dfrac{\sqrt{4n^2\pi^2c^2 - r^2l^2}}{2l}t\right) & n \geq 2 \end{cases}.$$

According to the principle of linear superposition, the solution to the PDE for $u(x,t)$ is a linear combination of all products $T_n(t)X_n(x)$ over all the eigenvalues.

$$\begin{aligned} u(x,t) = \left\{A_1 \exp\left[\left(-\frac{r}{2} + \frac{\sqrt{r^2l^2 - 4\pi^2c^2}}{2l}\right)t\right] + B_1 \exp\left[\left(-\frac{r}{2} - \frac{\sqrt{r^2l^2 - 4\pi^2c^2}}{2l}\right)t\right]\right\}\sin\frac{\pi x}{l} \\ + \sum_{n=2}^{\infty}\left[A_n \exp\left(-\frac{r}{2}t\right)\cos\left(\frac{\sqrt{4n^2\pi^2c^2 - r^2l^2}}{2l}t\right)\right. \\ \left. + B_n \exp\left(-\frac{r}{2}t\right)\sin\left(\frac{\sqrt{4n^2\pi^2c^2 - r^2l^2}}{2l}t\right)\right]\sin\frac{n\pi x}{l} \end{aligned}$$

Therefore,

$$\boxed{\begin{aligned} u(x,t) = \exp\left(-\frac{r}{2}t\right)\left[A_1 \exp\left(\frac{\sqrt{r^2l^2 - 4\pi^2c^2}}{2l}t\right) + B_1 \exp\left(-\frac{\sqrt{r^2l^2 - 4\pi^2c^2}}{2l}t\right)\right]\sin\frac{\pi x}{l} \\ + \sum_{n=2}^{\infty}\exp\left(-\frac{r}{2}t\right)\left[A_n \cos\left(\frac{\sqrt{4n^2\pi^2c^2 - r^2l^2}}{2l}t\right) + B_n \sin\left(\frac{\sqrt{4n^2\pi^2c^2 - r^2l^2}}{2l}t\right)\right]\sin\frac{n\pi x}{l}. \end{aligned}}$$

The final task is to use Fourier's method to express the coefficients, A_1, B_1, A_n, and B_n, in terms of the provided initial data, $\phi(x)$ and $\psi(x)$.

$$u(x,0) = (A_1 + B_1)\sin\frac{\pi x}{l} + \sum_{n=2}^{\infty} A_n \sin\frac{n\pi x}{l} = \phi(x)$$

Multiply both sides by $\sin \lambda_m x$, where m is a positive integer.

$$(A_1 + B_1)\sin\frac{\pi x}{l}\sin\frac{m\pi x}{l} + \sum_{n=2}^{\infty} A_n \sin\frac{n\pi x}{l}\sin\frac{m\pi x}{l} = \phi(x)\sin\frac{m\pi x}{l}$$

Integrate both sides with respect to x over the domain the PDE is defined.

$$\int_0^l \left[(A_1 + B_1)\sin\frac{\pi x}{l}\sin\frac{m\pi x}{l} + \sum_{n=2}^{\infty} A_n \sin\frac{n\pi x}{l}\sin\frac{m\pi x}{l}\right] dx = \int_0^l \phi(x)\sin\frac{m\pi x}{l}\,dx$$

Split up the integral into two and move the constants in front of them.

$$(A_1 + B_1)\int_0^l \sin\frac{\pi x}{l}\sin\frac{m\pi x}{l}\,dx + \sum_{n=2}^{\infty} A_n \int_0^l \sin\frac{n\pi x}{l}\sin\frac{m\pi x}{l}\,dx = \int_0^l \phi(x)\sin\frac{m\pi x}{l}\,dx$$

Since n and m are integers,

$$\int_0^l \sin\frac{n\pi x}{l}\sin\frac{m\pi x}{l}\,dx = \begin{cases} \frac{l}{2} & n = m \\ 0 & n \neq m \end{cases},$$

as can be verified with trigonometric identities. If $m = 1$, then every term in the infinite series is zero because n is never equal to 1.

$$\begin{aligned} m = 1: \quad (A_1 + B_1)\int_0^l \sin\frac{\pi x}{l}\sin\frac{\pi x}{l}\,dx &= \int_0^l \phi(x)\sin\frac{\pi x}{l}\,dx \\ (A_1 + B_1)\cdot\frac{l}{2} &= \int_0^l \phi(x)\sin\frac{\pi x}{l}\,dx \end{aligned}$$

$$A_1 + B_1 = \frac{2}{l}\int_0^l \phi(x)\sin\frac{\pi x}{l}\,dx \tag{1}$$

This is one equation for the two unknowns, A_1 and B_1. Now consider the case where $m \geq 2$. The first term on the left is then zero, and every term in the infinite series is zero except for the one when $n = m$.

$$\begin{aligned} m \geq 2: \quad \sum_{n=2}^{\infty} A_n \int_0^l \sin\frac{n\pi x}{l}\sin\frac{m\pi x}{l}\,dx &= \int_0^l \phi(x)\sin\frac{m\pi x}{l}\,dx \\ A_n \cdot \frac{l}{2} &= \int_0^l \phi(x)\sin\frac{n\pi x}{l}\,dx \end{aligned}$$

Therefore,

$$\boxed{A_n = \frac{2}{l}\int_0^l \phi(x)\sin\frac{n\pi x}{l}\,dx, \quad n \geq 2.}$$

In order to use the second initial condition, differentiate the solution for u with respect to t.

$$\begin{aligned}
u_t(x,t) = \Bigg\{ & A_1 \left(-\frac{r}{2} + \frac{\sqrt{r^2l^2 - 4\pi^2c^2}}{2l}\right) \exp\left[\left(-\frac{r}{2} + \frac{\sqrt{r^2l^2 - 4\pi^2c^2}}{2l}\right)t\right] \\
& + B_1 \left(-\frac{r}{2} - \frac{\sqrt{r^2l^2 - 4\pi^2c^2}}{2l}\right) \exp\left[\left(-\frac{r}{2} - \frac{\sqrt{r^2l^2 - 4\pi^2c^2}}{2l}\right)t\right]\Bigg\} \sin\frac{\pi x}{l} \\
& + \sum_{n=2}^{\infty} \left(-\frac{r}{2}\right) \exp\left(-\frac{r}{2}t\right) \left[A_n \cos\left(\frac{\sqrt{4n^2\pi^2c^2 - r^2l^2}}{2l}t\right) + B_n \sin\left(\frac{\sqrt{4n^2\pi^2c^2 - r^2l^2}}{2l}t\right)\right] \sin\frac{n\pi x}{l} \\
& + \sum_{n=2}^{\infty} \exp\left(-\frac{r}{2}t\right) \left[-A_n \frac{\sqrt{4n^2\pi^2c^2 - r^2l^2}}{2l} \sin\left(\frac{\sqrt{4n^2\pi^2c^2 - r^2l^2}}{2l}t\right)\right. \\
& \qquad \left. + B_n \frac{\sqrt{4n^2\pi^2c^2 - r^2l^2}}{2l} \cos\left(\frac{\sqrt{4n^2\pi^2c^2 - r^2l^2}}{2l}t\right)\right] \sin\frac{n\pi x}{l}
\end{aligned}$$

Now plug in $t = 0$ and set the result equal to $\psi(x)$.

$$\begin{aligned}
u_t(x,0) = & \left[A_1 \left(-\frac{r}{2} + \frac{\sqrt{r^2l^2 - 4\pi^2c^2}}{2l}\right) + B_1 \left(-\frac{r}{2} - \frac{\sqrt{r^2l^2 - 4\pi^2c^2}}{2l}\right)\right] \sin\frac{\pi x}{l} \\
& + \sum_{n=2}^{\infty} \left(-\frac{r}{2}\right) A_n \sin\frac{n\pi x}{l} + \sum_{n=2}^{\infty} B_n \frac{\sqrt{4n^2\pi^2c^2 - r^2l^2}}{2l} \sin\frac{n\pi x}{l} = \psi(x)
\end{aligned}$$

Multiply both sides by $\sin\lambda_m x$, where m is a positive integer.

$$\begin{aligned}
& \left[A_1 \left(-\frac{r}{2} + \frac{\sqrt{r^2l^2 - 4\pi^2c^2}}{2l}\right) + B_1 \left(-\frac{r}{2} - \frac{\sqrt{r^2l^2 - 4\pi^2c^2}}{2l}\right)\right] \sin\frac{\pi x}{l} \sin\frac{m\pi x}{l} \\
& + \sum_{n=2}^{\infty} \left(-\frac{r}{2}\right) A_n \sin\frac{n\pi x}{l} \sin\frac{m\pi x}{l} + \sum_{n=2}^{\infty} B_n \frac{\sqrt{4n^2\pi^2c^2 - r^2l^2}}{2l} \sin\frac{n\pi x}{l} \sin\frac{m\pi x}{l} = \psi(x) \sin\frac{m\pi x}{l}
\end{aligned}$$

Integrate both sides with respect to x over the domain the PDE is defined.

$$\begin{aligned}
\int_0^l \Bigg\{ & \left[A_1 \left(-\frac{r}{2} + \frac{\sqrt{r^2l^2 - 4\pi^2c^2}}{2l}\right) + B_1 \left(-\frac{r}{2} - \frac{\sqrt{r^2l^2 - 4\pi^2c^2}}{2l}\right)\right] \sin\frac{\pi x}{l} \sin\frac{m\pi x}{l} \\
& + \sum_{n=2}^{\infty} \left(-\frac{r}{2}\right) A_n \sin\frac{n\pi x}{l} \sin\frac{m\pi x}{l} \\
& \qquad + \sum_{n=2}^{\infty} B_n \frac{\sqrt{4n^2\pi^2c^2 - r^2l^2}}{2l} \sin\frac{n\pi x}{l} \sin\frac{m\pi x}{l} \Bigg\} \, dx = \int_0^l \psi(x) \sin\frac{m\pi x}{l} \, dx
\end{aligned}$$

Split up the integral into three and move the constants in front of them.

$$\begin{aligned}
& \left[A_1 \left(-\frac{r}{2} + \frac{\sqrt{r^2l^2 - 4\pi^2c^2}}{2l}\right) + B_1 \left(-\frac{r}{2} - \frac{\sqrt{r^2l^2 - 4\pi^2c^2}}{2l}\right)\right] \int_0^l \sin\frac{\pi x}{l} \sin\frac{m\pi x}{l} \, dx \\
& + \sum_{n=2}^{\infty} \left(-\frac{r}{2}\right) A_n \int_0^l \sin\frac{n\pi x}{l} \sin\frac{m\pi x}{l} \, dx \\
& \qquad + \sum_{n=2}^{\infty} B_n \frac{\sqrt{4n^2\pi^2c^2 - r^2l^2}}{2l} \int_0^l \sin\frac{n\pi x}{l} \sin\frac{m\pi x}{l} \, dx = \int_0^l \psi(x) \sin\frac{m\pi x}{l} \, dx
\end{aligned}$$

If $m = 1$, then every term in the infinite series is zero because n is never equal to 1.

$$m = 1: \quad \left[A_1\left(-\frac{r}{2} + \frac{\sqrt{r^2l^2 - 4\pi^2c^2}}{2l}\right) + B_1\left(-\frac{r}{2} - \frac{\sqrt{r^2l^2 - 4\pi^2c^2}}{2l}\right)\right]\int_0^l \sin\frac{\pi x}{l}\sin\frac{\pi x}{l}\,dx = \int_0^l \psi(x)\sin\frac{\pi x}{l}\,dx$$

$$\left[A_1\left(-\frac{r}{2} + \frac{\sqrt{r^2l^2 - 4\pi^2c^2}}{2l}\right) + B_1\left(-\frac{r}{2} - \frac{\sqrt{r^2l^2 - 4\pi^2c^2}}{2l}\right)\right]\cdot\frac{l}{2} = \int_0^l \psi(x)\sin\frac{\pi x}{l}\,dx$$

$$A_1\left(-\frac{r}{2} + \frac{\sqrt{r^2l^2 - 4\pi^2c^2}}{2l}\right) + B_1\left(-\frac{r}{2} - \frac{\sqrt{r^2l^2 - 4\pi^2c^2}}{2l}\right) = \frac{2}{l}\int_0^l \psi(x)\sin\frac{\pi x}{l}\,dx \qquad (2)$$

This is another equation for the two unknowns, A_1 and B_1. Solving the system of equations, (1) and (2), for A_1 and B_1 gives us

$$A_1 = \frac{-\left(-\frac{r}{2} - \frac{\sqrt{r^2l^2 - 4\pi^2c^2}}{2l}\right)\frac{2}{l}\int_0^l \phi(x)\sin\frac{\pi x}{l}\,dx + \frac{2}{l}\int_0^l \psi(x)\sin\frac{\pi x}{l}\,dx}{\left(-\frac{r}{2} + \frac{\sqrt{r^2l^2 - 4\pi^2c^2}}{2l}\right) - \left(-\frac{r}{2} - \frac{\sqrt{r^2l^2 - 4\pi^2c^2}}{2l}\right)}$$

$$B_1 = \frac{\left(-\frac{r}{2} + \frac{\sqrt{r^2l^2 - 4\pi^2c^2}}{2l}\right)\frac{2}{l}\int_0^l \phi(x)\sin\frac{\pi x}{l}\,dx - \frac{2}{l}\int_0^l \psi(x)\sin\frac{\pi x}{l}\,dx}{\left(-\frac{r}{2} + \frac{\sqrt{r^2l^2 - 4\pi^2c^2}}{2l}\right) - \left(-\frac{r}{2} - \frac{\sqrt{r^2l^2 - 4\pi^2c^2}}{2l}\right)}.$$

Simplify these answers.

$$A_1 = \frac{\frac{2}{l}\int_0^l \left[\left(\frac{r}{2} + \frac{\sqrt{r^2l^2 - 4\pi^2c^2}}{2l}\right)\phi(x) + \psi(x)\right]\sin\frac{\pi x}{l}\,dx}{\frac{\sqrt{r^2l^2 - 4\pi^2c^2}}{l}}$$

$$B_1 = \frac{\frac{2}{l}\int_0^l \left[\left(-\frac{r}{2} + \frac{\sqrt{r^2l^2 - 4\pi^2c^2}}{2l}\right)\phi(x) - \psi(x)\right]\sin\frac{\pi x}{l}\,dx}{\frac{\sqrt{r^2l^2 - 4\pi^2c^2}}{l}}$$

Therefore,

$$\boxed{A_1 = \frac{2}{\sqrt{r^2l^2 - 4\pi^2c^2}}\int_0^l \left[\left(\frac{r}{2} + \frac{\sqrt{r^2l^2 - 4\pi^2c^2}}{2l}\right)\phi(x) + \psi(x)\right]\sin\frac{\pi x}{l}\,dx}$$

and

$$\boxed{B_1 = \frac{2}{\sqrt{r^2l^2 - 4\pi^2c^2}}\int_0^l \left[\left(-\frac{r}{2} + \frac{\sqrt{r^2l^2 - 4\pi^2c^2}}{2l}\right)\phi(x) - \psi(x)\right]\sin\frac{\pi x}{l}\,dx.}$$

Now consider the case where $m \geq 2$. The first term on the left is then zero, and every term in the infinite series is zero except for the ones when $n = m$.

$$m \geq 2: \quad \sum_{n=2}^{\infty}\left(-\frac{r}{2}\right) A_n \int_0^l \sin\frac{n\pi x}{l}\sin\frac{m\pi x}{l}\,dx + \sum_{n=2}^{\infty} B_n \frac{\sqrt{4n^2\pi^2c^2 - r^2l^2}}{2l}\int_0^l \sin\frac{n\pi x}{l}\sin\frac{m\pi x}{l}\,dx = \int_0^l \psi(x)\sin\frac{m\pi x}{l}\,dx$$

$$\left(-\frac{r}{2}\right) A_n \cdot \frac{l}{2} + B_n \frac{\sqrt{4n^2\pi^2c^2 - r^2l^2}}{2l}\cdot\frac{l}{2} = \int_0^l \psi(x)\sin\frac{n\pi x}{l}\,dx$$

Multiply both sides by $2/l$.

$$\left(-\frac{r}{2}\right) A_n + B_n \frac{\sqrt{4n^2\pi^2c^2 - r^2l^2}}{2l} = \frac{2}{l}\int_0^l \psi(x)\sin\frac{n\pi x}{l}\,dx$$

Bring the term with A_n to the right side.

$$B_n \frac{\sqrt{4n^2\pi^2c^2 - r^2l^2}}{2l} = \frac{r}{2}A_n + \frac{2}{l}\int_0^l \psi(x)\sin\frac{n\pi x}{l}\,dx$$

Substitute the expression for A_n.

$$B_n \frac{\sqrt{4n^2\pi^2c^2 - r^2l^2}}{2l} = \frac{r}{2}\cdot\frac{2}{l}\int_0^l \phi(x)\sin\frac{n\pi x}{l}\,dx + \frac{2}{l}\int_0^l \psi(x)\sin\frac{n\pi x}{l}\,dx$$

Combine the integrals on the right side.

$$B_n \frac{\sqrt{4n^2\pi^2c^2 - r^2l^2}}{2l} = \frac{2}{l}\int_0^l \left[\frac{r}{2}\phi(x) + \psi(x)\right]\sin\frac{n\pi x}{l}\,dx$$

Therefore,

$$\boxed{B_n = \frac{4}{\sqrt{4n^2\pi^2c^2 - r^2l^2}}\int_0^l \left[\frac{r}{2}\phi(x) + \psi(x)\right]\sin\frac{n\pi x}{l}\,dx, \quad n \geq 2.}$$