Exercise 6

Separate the variables for the equation $tu_t = u_{xx} + 2u$ with the boundary conditions $u(0, t) = u(\pi, t) = 0$. Show that there are an infinite number of solutions that satisfy the initial condition $u(x, 0) = 0$. So uniqueness is false for this equation!

Solution

Because the PDE and its boundary conditions are linear and homogeneous, the method of separation of variables can be applied to solve it. Assume a product solution of the form, $u(x, t) = X(x)T(t)$, and plug it into the PDE

$$tu_t = u_{xx} + 2u \quad \rightarrow \quad tXT' = X''T + 2XT$$

and the boundary conditions.

$$\begin{aligned} u(0, t) &= X(0)T(t) = 0 \qquad &\rightarrow \qquad X(0) = 0 \\ u(\pi, t) &= X(\pi)T(t) = 0 \qquad &\rightarrow \qquad X(\pi) = 0 \end{aligned}$$

Separate variables now.

$$tXT' - 2XT = X''T$$

Divide both sides by XT.

$$\frac{tT'}{T} - 2 = \frac{X''}{X}$$

We have a function of t on the left side and a function of x on the right side. The only way both functions can be equal is if they are equal to a constant.

$$\frac{tT'}{T} - 2 = \frac{X''}{X} = k$$

Values of k for which $X(0) = 0$ and $X(\pi) = 0$ are satisfied are called the eigenvalues, and the nontrivial functions $X(x)$ associated with them are called the eigenfunctions.

Determination of Positive Eigenvalues: $k = \mu^2$

Assuming k is positive, the differential equation for X becomes

$$\frac{X''}{X} = \mu^2$$

Multiply both sides by X.

$$X'' = \mu^2 X$$

The general solution can be written in terms of hyperbolic sine and hyperbolic cosine.

$$X(x) = C_1 \cosh \mu x + C_2 \sinh \mu x$$

Now use the boundary conditions to determine C_1 and C_2.

$$\begin{aligned} X(0) &= C_1 = 0 \\ X(\pi) &= C_1 \cosh \mu\pi + C_2 \sinh \mu\pi = 0 \end{aligned}$$

We see that $C_1 = 0$ and $C_2 = 0$. Hence, only the trivial solution $X(x) = 0$ results from considering positive values for k, and there are no positive eigenvalues.

Determination of the Zero Eigenvalue: $k = 0$

Assuming k is zero, the differential equation for X becomes

$$\frac{X''}{X} = 0.$$

Multiply both sides by X.

$$X'' = 0$$

The general solution is a linear function.

$$X(x) = C_3 x + C_4$$

Now use the boundary conditions to determine C_3 and C_4.

$$\begin{aligned} X(0) &= C_4 = 0 \\ X(\pi) &= C_3 \pi + C_4 = 0 \end{aligned}$$

We see that $C_3 = 0$ and $C_4 = 0$. Hence, only the trivial solution $X(x) = 0$ results from considering $k = 0$, and zero is not an eigenvalue.

Determination of Negative Eigenvalues: $k = -\lambda^2$

Assuming k is negative, the differential equation for X becomes

$$\frac{X''}{X} = -\lambda^2.$$

Multiply both sides by X.

$$X'' = -\lambda^2 X$$

The general solution can be written in terms of sine and cosine.

$$X(x) = C_5 \cos \lambda x + C_6 \sin \lambda x$$

Now use the boundary conditions to determine C_5 and C_6.

$$\begin{aligned} X(0) &= C_5 = 0 \\ X(\pi) &= C_5 \cos \lambda \pi + C_6 \sin \lambda \pi = 0 \end{aligned}$$

The second equation simplifies to $C_6 \sin \lambda \pi = 0$. To avoid getting the trivial solution, we insist that $C_6 \neq 0$. Doing so yields an equation for the eigenvalues.

$$\sin \lambda \pi = 0$$

Solve for $\lambda \pi$.

$$\lambda \pi = n\pi, \quad n = 1, 2, \ldots$$

So then

$$\lambda = \lambda_n = n, \quad n = 1, 2, \ldots.$$

The eigenfunctions associated with these eigenvalues are

$$X(x) = C_6 \sin \lambda x \quad \to \quad X_n(x) = \sin \lambda_n x, \quad n = 1, 2, \ldots.$$

Now solve the differential equation for $T(t)$.

$$\frac{tT'}{T} - 2 = -\lambda^2$$

Bring λ^2 to the left side and multiply both sides by T.

$$tT' + (\lambda^2 - 2)T = 0$$

This is an equidimensional ODE for T, so the solution is of the form

$$T = t^r.$$

Substitute this into the ODE to determine r.

$$t(rt^{r-1}) + (\lambda^2 - 2)t^r = 0$$

$$rt^r + (\lambda^2 - 2)t^r = 0$$

Divide both sides by t^r.

$$r + (\lambda^2 - 2) = 0$$

This means that

$$r = 2 - \lambda^2,$$

so

$$T(t) = C_7 t^{2-\lambda^2} \quad \to \quad T_n(t) = t^{2-\lambda_n^2}.$$

According to the principle of linear superposition, the solution to the PDE for $u(x,t)$ is a linear combination of all products $T_n(t)X_n(x)$ over all the eigenvalues.

$$u(x,t) = \sum_{n=1}^{\infty} A_n t^{2-\lambda_n^2} \sin \lambda_n x$$

Thus,

$$u(x,t) = \sum_{n=1}^{\infty} A_n t^{2-n^2} \sin nx.$$

The coefficient A_n is determined by using the initial condition, $u(x,0) = 0$.

$$u(x,0) = \sum_{n=1}^{\infty} A_n (0)^{2-n^2} \sin nx = 0.$$

A_n can be anything because it's multiplied by something that makes the whole left side equal to 0. Therefore, if $u(x,0) = 0$, then there are an infinite number of solutions.