

Exercise 4

Consider diffusion inside an enclosed circular tube. Let its length (circumference) be $2l$. Let x denote the arc length parameter where $-l \leq x \leq l$. Then the concentration of the diffusing substance satisfies

$$u_t = ku_{xx} \quad \text{for } -l \leq x \leq l$$
$$u(-l, t) = u(l, t) \quad \text{and} \quad u_x(-l, t) = u_x(l, t).$$

These are called *periodic boundary conditions*.

(a) Show that the eigenvalues are $\lambda = (n\pi/l)^2$ for $n = 0, 1, 2, 3, \dots$

(b) Show that the concentration is

$$u(x, t) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} \left(A_n \cos \frac{n\pi x}{l} + B_n \sin \frac{n\pi x}{l} \right) e^{-n^2\pi^2 kt/l^2}.$$