Exercise 1

Find the eigenvalues graphically for the boundary conditions

$$X(0) = 0, \qquad X'(l) + aX(l) = 0.$$

Assume that $a \neq 0$.

Solution

The aim here is to determine the values of λ (eigenvalues) in the ODE,

$$-X'' = \lambda X,$$

that satisfy the provided boundary conditions. This problem comes about as a result of separating variables in a PDE.

Determination of Positive Eigenvalues: $\lambda = \mu^2$

Assuming λ is positive, the differential equation for X becomes

$$X'' = -\mu^2 X.$$

The general solution can be written in terms of sine and cosine.

$$X(x) = C_1 \cos \mu x + C_2 \sin \mu x$$

Now use the boundary conditions to determine C_1 and C_2.

$$\begin{aligned} x(0) = 0 \quad &\rightarrow \quad C_1 = 0 \\ X'(l) + aX(l) = 0 \quad &\rightarrow \quad \mu(-C_1 \sin \mu l + C_2 \cos \mu l) + a(C_1 \cos \mu l + C_2 \sin \mu l) = 0 \end{aligned}$$

From the second condition we see that C_2 cancels out, leaving a transcendental equation for μ.

$$\mu \cos \mu l + a \sin \mu l = 0$$

Divide both sides by $a \cos \mu l$.

$$\frac{\mu}{a} + \tan \mu l = 0$$

$$-\frac{\mu}{a} = \tan \mu l$$

In order to group the constants together, make the substitution $m = \mu l$. Then $\mu = m/l$.

$$-\frac{m}{al} = \tan m$$

The values of m that solve the equation are the m-coordinates of the intersections in the graphs of these two functions. Since the eigenvalues are given by

$$\lambda = \mu^2 = \left(\frac{m}{l}\right)^2$$

and both functions, $-m/(al)$ and $\tan m$, are odd, negative values of m give redundant values for λ. The value of al influences the locations of the intersections as illustrated in the following figures.

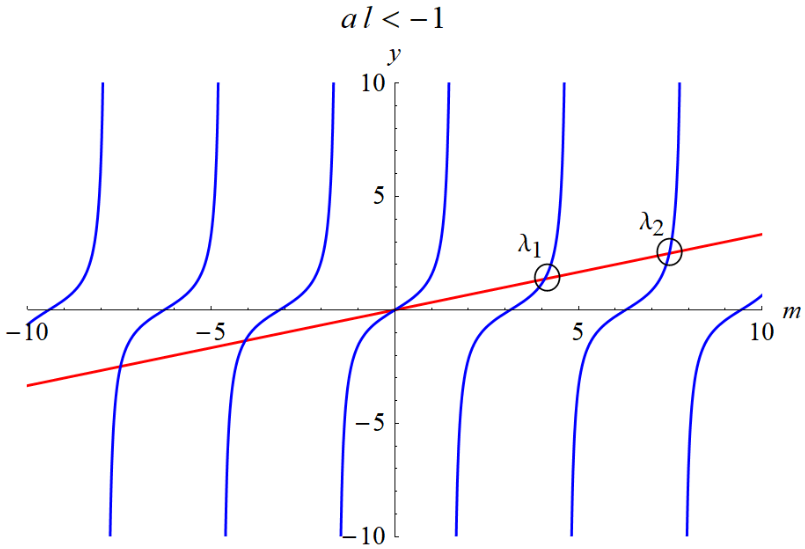

Figure 1: This is a plot of the line (in red) and the tangent function (in blue) for $al < -1$.

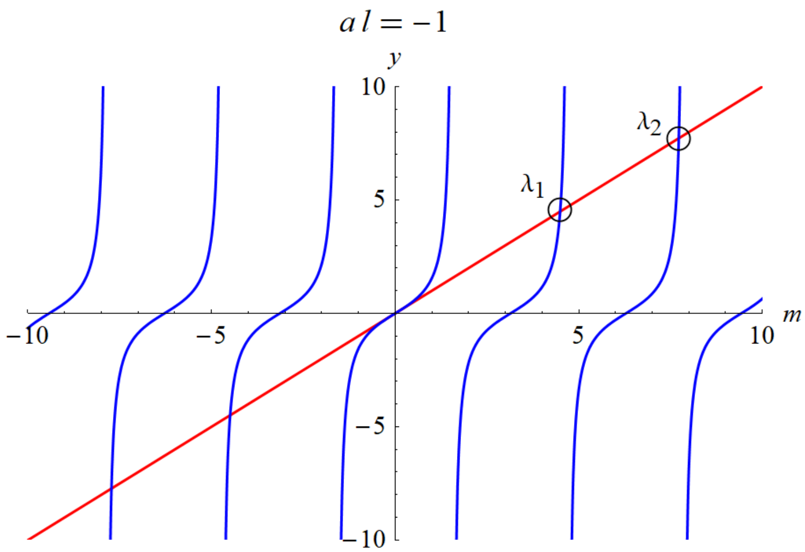

Figure 2: This is a plot of the line (in red) and the tangent function (in blue) for $al = -1$.

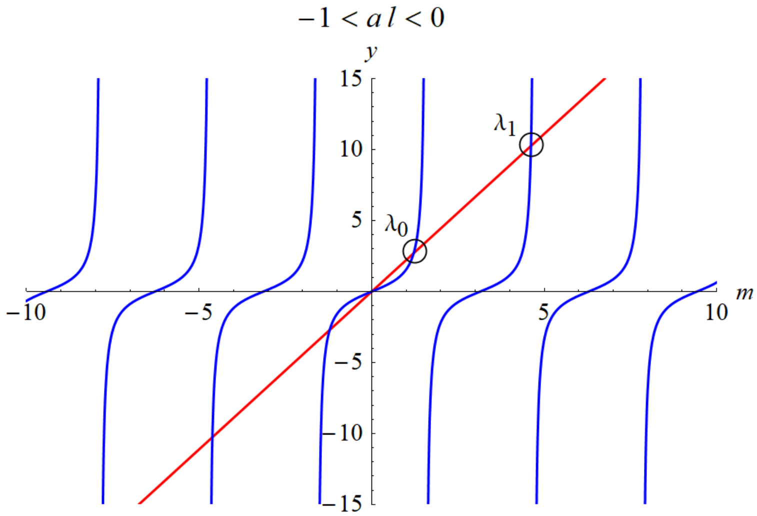

Figure 3: This is a plot of the line (in red) and the tangent function (in blue) for $-1 < al < 0$. Note that in this case there is an intersection on the curve of tangent going through the origin, which gives an extra eigenvalue λ_0.

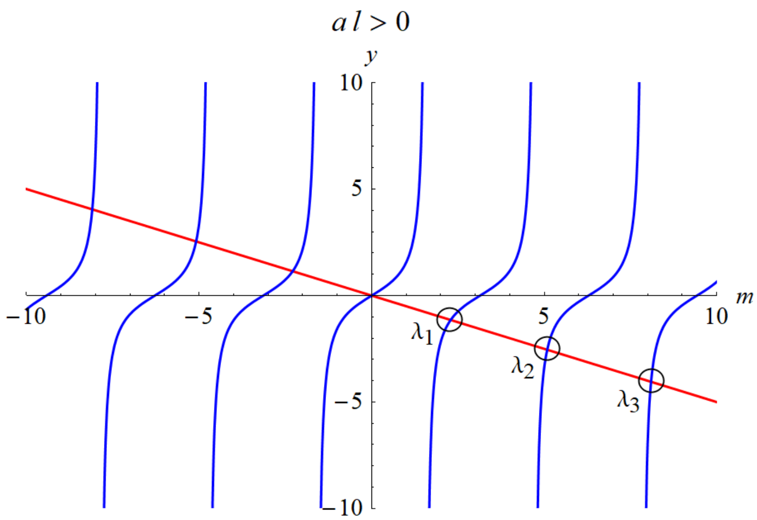

Figure 4: This is a plot of the line (in red) and the tangent function (in blue) for $al > 0$. The extra eigenvalue λ_0 is gone in this case.

Determination of the Zero Eigenvalue: $\lambda = 0$

Assuming λ is zero, the differential equation for X becomes

$$X'' = 0.$$

The general solution is a linear function.

$$X(x) = C_3 x + C_4$$

Apply the boundary conditions here to determine C_3 and C_4.

$$\begin{aligned} x(0) = 0 \quad &\rightarrow \quad C_4 = 0 \\ X'(l) + aX(l) = 0 \quad &\rightarrow \quad C_3 + a(C_3 l + C_4) = 0 \end{aligned}$$

The second equation simplifies to

$$C_3(1 + al) = 0,$$

so $C_3 = 0$. The trivial solution is obtained for $X(x)$, so zero is not an eigenvalue.

Determination of Negative Eigenvalues: $\lambda = -\eta^2$

Assuming η is positive, the differential equation for X becomes

$$X'' = \eta^2 X.$$

The general solution can be written in terms of hyperbolic sine and hyperbolic cosine.

$$X(x) = C_5 \cosh \eta x + C_6 \sinh \eta x$$

Now use the boundary conditions to determine C_5 and C_6.

$$\begin{aligned} x(0) = 0 \quad &\rightarrow \quad C_5 = 0 \\ X'(l) + aX(l) = 0 \quad &\rightarrow \quad \eta(C_5 \sinh \eta l + C_6 \cosh \eta l) + a(C_5 \cosh \eta l + C_6 \sinh \eta l) = 0 \end{aligned}$$

From the second condition we see that C_6 cancels out, leaving a transcendental equation for η.

$$\eta \cosh \eta l + a \sinh \eta l = 0$$

Divide both sides by $a \cosh \eta l$.

$$\frac{\eta}{a} + \tanh \eta l = 0$$

$$-\frac{\eta}{a} = \tanh \eta l$$

In order to group the constants together, make the substitution $n = \eta l$. Then $\eta = n/l$.

$$-\frac{n}{al} = \tanh n$$

The values of n that solve the equation are the n-coordinates of the intersections in the graphs of these two functions. Since the eigenvalues are given by

$$\lambda = -\eta^2 = \left(\frac{n}{l}\right)^2$$

and both functions, $-n/(al)$ and $\tanh n$, are odd, negative values of n give redundant values for λ. The value of al influences the locations of the intersections as illustrated in the following figures.

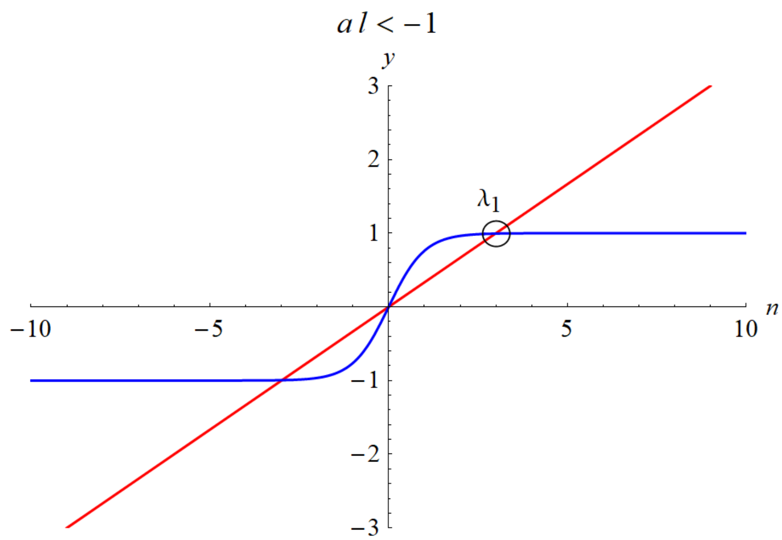

Figure 5: This is a plot of the line (in red) and the hyperbolic tangent function (in blue) for $al < -1$. There is one eigenvalue η_1.

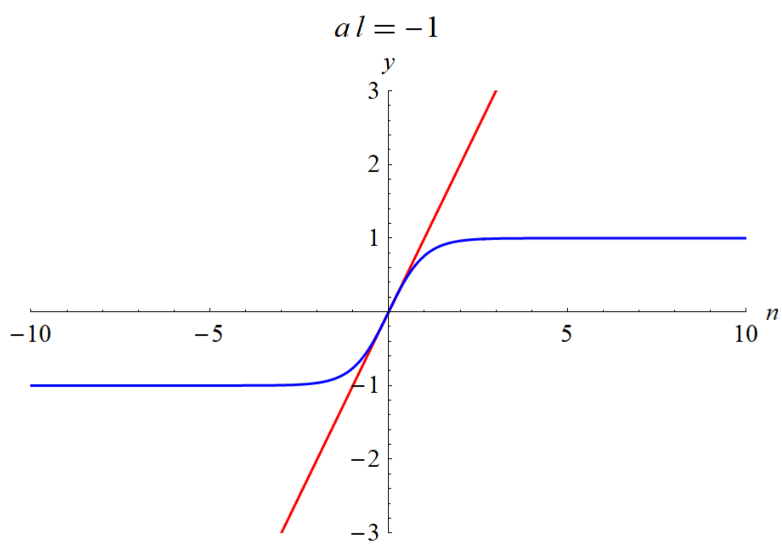

Figure 6: This is a plot of the line (in red) and the hyperbolic tangent function (in blue) for $al = -1$. The eigenvalue η_1 is gone in this case.

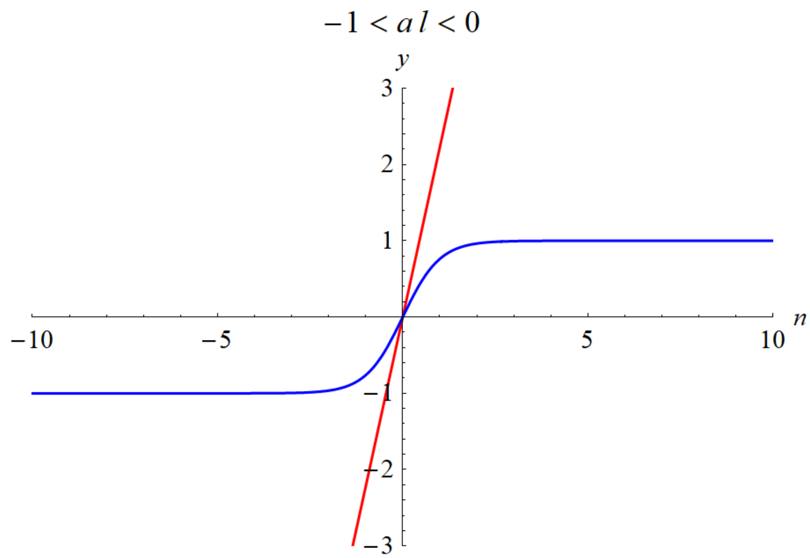

Figure 7: This is a plot of the line (in red) and the hyperbolic tangent function (in blue) for $-1 < al < 0$.

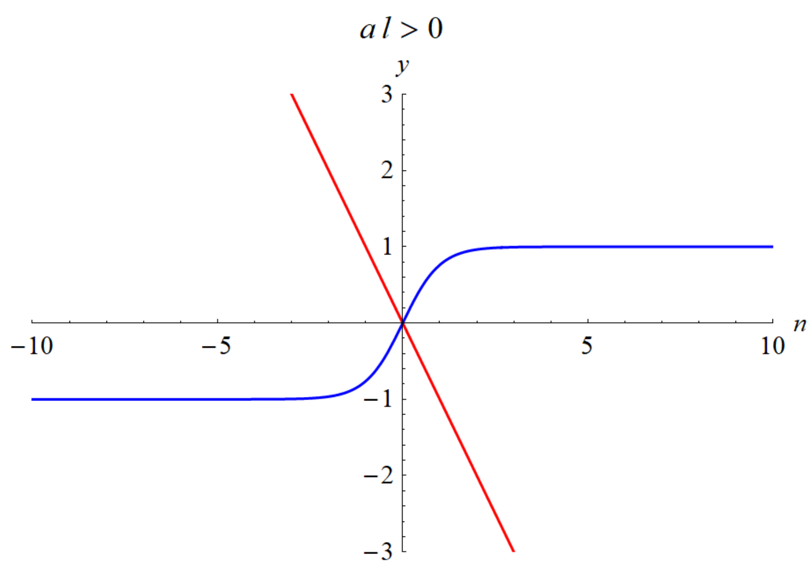

Figure 8: This is a plot of the line (in red) and the hyperbolic tangent function (in blue) for $al > 0$.

In conclusion, the positive eigenvalues of the ODE, $-X'' = \lambda X$, subject to the boundary conditions, $X(0) = 0$ and $X'(l) + aX(l) = 0$, are

$$\lambda = \frac{m^2}{l^2},$$

where m is a solution to the equation

$$-\frac{m}{al} = \tan m.$$

The eigenfunctions associated with these positive eigenvalues are

$$X(x) = C_2 \sin \mu x \quad \rightarrow \quad X_k(x) = \sin \frac{m_k x}{l},$$

where k is an index going over all the positive eigenvalues. The negative eigenvalues of the ODE subject to the boundary conditions are

$$\lambda = -\frac{n^2}{l^2},$$

where n is a solution to the equation

$$-\frac{n}{al} = \tanh n.$$

The eigenfunctions associated with these negative eigenvalues are

$$X(x) = C_6 \sinh \eta x \quad \rightarrow \quad X_1(x) = \sinh \frac{n_1 x}{l}.$$

Since there's only one eigenvalue depending on al as indicated by the graphs, there's only one possible eigenfunction.